Vol. 1

MATHEMATICS - I

(Calculus and Linear Algebra)

For Non-Computer Science Engineering Branches

Reena Garg

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Vol-1 ***For Non-Computer Science Engineering Branches*** *by* Reena Garg

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FOREWORD

Engineering has played a very significant role in the progress and expansion of mankind and society for centuries. Engineering ideas that originated in the Indian subcontinent have had a thoughtful impact on the world.

All India Council for Technical Education (AICTE) had always been at the forefront of assisting Technical students in every possible manner since its inception in 1987. The goal of AICTE has been to promote quality Technical Education and thereby take the industry to a greater heights and ultimately turn our dear motherland India into a Modern Developed Nation. It will not be inept to mention here that Engineers are the backbone of the modern society - better the engineers, better the industry, and better the industry, better the country.

NEP 2020 envisages education in regional languages to all, thereby ensuring that each and every student becomes capable and competent enough and is in a position to contribute towards the national growth and development.

One of the spheres where AICTE had been relentlessly working from last few years was to provide high-quality moderately priced books of International standard prepared in various regional languages to all it's Engineering students. These books are not only prepared keeping in mind it's easy language, real life examples, rich contents and but also the industry needs in this everyday changing world. These books are as per AICTE Model Curriculum of Engineering & Technology – 2018.

Eminent Professors from all over India with great knowledge and experience have written these books for the benefit of academic fraternity. AICTE is confident that these books with their rich contents will help technical students master the subjects with greater ease and quality.

AICTE appreciates the hard work of the original authors, coordinators and the translators for their endeavour in making these Engineering subjects more lucid.

(Anil D. Sahasrabudhe)

Acknowledgement

The author grateful to AICTE for their meticulous planning and execution to publish the technical book for Engineering and Technology students.

I sincerely acknowledge the valuable contributions of the reviewer of the book Prof. Garima Singh, for making it students' friendly and giving a better shape in an artistic manner.

This book is an outcome of various suggestions of AICTE members, experts and authors who shared their opinion and thoughts to further develop the engineering education in our country.

It is also with great honour that I state that this book is aligned to the AICTE Model Curriculum and in line with the guidelines of National Education Policy (NEP) -2020. Towards promoting education in regional languages, this book is being translated in scheduled Indian regional languages.

Acknowledgements are due to the contributors and different workers in this field whose published books, review articles, papers, photographs, footnotes, references and other valuable information enriched us at the time of writing the book.

Finally, I like to express my sincere thanks to the publishing house, M/s. Khanna Book Publishing Company Private Limited, New Delhi, whose entire team was always ready to cooperate on all the aspects of publishing to make it a wonderful experience.

Reena Garg

Preface

Mathematics is a necessary avenue to scientific knowledge which opens new vistas of mental ability. Engineering mathematics offers a balance of theory and practice, which is intellectually stimulating. Learning the craft of applying mathematics to real world problems allow an Engineering student to find the solutions of the problem

Calculus and Linear Algebra is intended mainly for undergraduate students of B.Tech of 21st century with the aim to provide a sound understanding in the subject of Mathematics. This book is strictly aligned with AICTE model curriculum incorporating student centric and self-learning activities as per New National Education Policy based on **OBE** and **Bloom Taxonomy**. The topics are well organized to create interest among readers to study and apply the mathematical tools in engineering and science disciplines. The book mainly emphasizes on the practical applications of the concepts discussed in the units which will help the students to incorporate a deliberate focus on problem - solving skills.

The book consists of 5 units. For more understanding of the topic, a good number of relatively competitive problems are given at the end of each unit in the form of **short questions, HOTS, assignments, MCQs** and **know more**. **Practical/Projects/Activity** also given in each unit for enhancing the student's capability and to increase the feeling of team work. To clarify the subject, the text has been supplemented through **Notes, Observations** and **Remarks**. An attempt has been made to explain the topics through maximum use of geometries wherever possible.

Unit-1 deals with the application of derivatives, curvature, definite and improper integrals, Beta-Gamma functions with their properties,

Unit-2 is concerned to find the solution by using Rolle's theorem, Mean value theorem, Taylor's and Maclaurin's theorems, L'Hospital Rule and Maxima-minima for one variable.

Unit-3 deals with convergence and divergence of sequence and series along with various tests for convergence, Taylor's and Maclaurin's series, power series, Fourier series along with examples.

Unit-4 focuses on limit, continuity, derivatives, tangent plane and normal line, maxima-minima for two variables, Lagrange method of multipliers, gradient, curl and divergence; their physical interpretation along with numerical problems.

Unit-5 discusses eigen values, eigenvectors, diagonalization, theorems based on symmetric and skew-symmetric matrices, Cayley –Hamilton theorem etc.

Mathematics is a subject that can be mastered only through hard work and practice. Practice is the only key word in the learning process of mathematics.

I hope this book will meet the requirements and expectations of all the engineering students. Although every care has been taken to avoid misprints and mistakes, yet it is difficult to claim perfection. I will gratefully receive and acknowledge every comment and suggestions from the teachers and the students leading to improvements in the text as well as in solved examples.

Reena Garg

Outcome Based Education

For the implementation of an outcome based education the first requirement is to develop an outcome based curriculum and incorporate an outcome based assessment in the education system. By going through outcome based assessments evaluators will be able to evaluate whether the students have achieved the outlined standard, specific and measurable outcomes. With the proper incorporation of outcome based education there will be a definite commitment to achieve a minimum standard for all learners without giving up at any level. At the end of the programme running with the aid of outcome based education, a student will be able to arrive at the following outcomes:

PO-1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO-2. **Problem analysis:** Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO-3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO-4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO-5. **Modern tool usage**1 Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO-6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO-7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO-8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO-9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO-10. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO-11. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO-12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

Course Outcomes

After completion of the course the students will be able to:

CO-1: Apply differential calculus to notion of curvature; centre of curvature and to evaluate improper integrals using correct mathematical limit notation. Apart from these applications they will have a basic understanding of Beta and Gamma functions based on integral calculus.

CO-2: Examine the behaviour of function for a given interval and expansion of trigonometric and transcendental functions, learn about indeterminate forms, their occurrence in problems and to calculate limits in it by repeated use of L′ Hospital rule.

CO-3: Determine the boundedness, convergence and divergence of sequence using concept of limit at infinity; interpret the concept of a series as the sum of a sequence and apply various tests to decide whether infinite series converge or diverge including alternating series; learn the concept of power series and Fourier series.

CO-4: Compare and contrast the ideas of continuity and differentiability; acquire the concept of finding partial derivatives and develop competency in finding the extreme values of functions of several independent variables; explain the concept of vector differentiation and demonstrate the use of vector identities in solving problems related to gradient, divergence and curl.

CO-5: Apply elementary transformations to reduce the matrix to Echelon form to find rank; interpret the solutions of system of linear equations; compute eigenvalues and eigenvectors of a matrix; become familiar with the concept of diagonalization and Cayley-Hamilton theorem and its uses.

Mapping of Course Outcomes with Programme Outcomes to be done according to the matrix given below:

Course Outcome	Expected Mapping with Programme Outcomes (1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)											
	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7	PO-8	PO-9	PO-10	PO-11	PO-12
CO-1	3	3	2	2	-	-	-	-	-	-	-	-
CO-2	2	3	2	2	-	-	-	-	-	-	-	1
CO-3	3	3	3	3	2	1	1	-	-	1	1	1
CO-4	2	2	3	2	1	-	-	-	-	-	-	-
CO-5	3	3	2	1	2	1	1	-	-	1	1	-

Abbreviations and Symbols

SYMBOLS AND FORMULAE

1. Number System

N	–	set of natural numbers
$\mathbb{Z}$	–	set of integers
Q	–	set of rational numbers
I	–	set of irrational numbers
$\mathbb{R}$	–	set of real numbers
C	–	set of complex numbers
R^n	–	set of n-tuples

2. Greek Letters

α	–	alpha
β	–	beta
γ	–	gamma
Γ	–	capital gamma
δ	–	delta
Δ	–	capital delta
ε	–	epsilon
ι	–	iota
θ	–	theta
λ	–	lambda
μ	–	mu
ϕ	–	phi
ψ	–	psi
η	–	eta
π	–	pi
ρ	–	rho
κ	–	kappa

3. Notation in sets

$\in$	–	belongs to
$\notin$	–	not belongs to
$\cup$	–	Union
$\cap$	–	Intersection
( )	–	open interval
[ ]	–	close interval
$\subseteq$	–	subset
$\not\subseteq$	–	not subset
$\subset$	–	proper subset
$\not\subset$	–	not a proper subset
$\supset$	–	superset
{ }	–	set
ϕ	–	empty set
$<$	–	strictly less than
$>$	–	strictly greater than
$\leq$	–	less than or equal to
$\geq$	–	greater than or equal to

4. Some Other Useful Symbols

$\sim$	–	equivalent to
$\leftrightarrow$	–	interchange
∞	–	infinity
$\int$	–	integral
!	–	factorial
$\Rightarrow$	–	implies
$\forall$	–	for all
$\Leftrightarrow$	–	implies and implied by
$\Vert\ \Vert$	–	norm
$\vert\ \vert$	–	modulus
:	–	colon
;	–	semicolon

$[A : B]$ or $[A/B]$ – Augmented Matrix

5. Nature of Roots of an Quadratic equations

If $ax^2 + bx + c = 0$ is quadratic, then

(*a*) its roots are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(*b*) the sum of the roots is equal to $-b/a$

(*c*) product of the roots is equal to c/a

(*d*) $b^2 - 4ac = 0 \quad \Rightarrow$ the roots are equal

(*e*) $b^2 - 4ac > 0 \quad \Rightarrow$ the roots are real and distinct

(*f*) $b^2 - 4ac < 0 \quad \Rightarrow$ the roots are complex

(*g*) If $b^2 - 4ac$ is a perfect square, then the roots are rational.

6. Properties of Logarithm

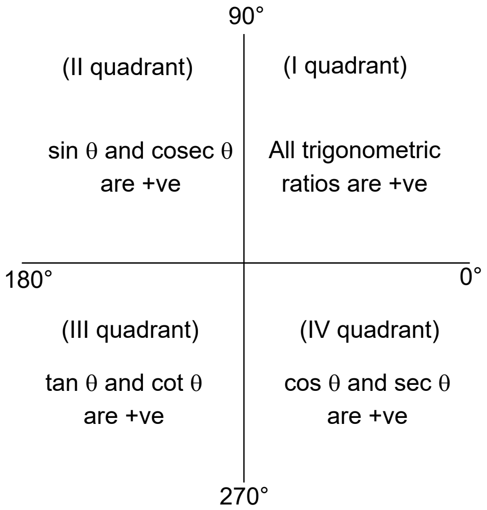
(*a*) $\log_a 1 = 0, \log_a 0 = -\infty$ for $a > 1$,
$\log_a a = 1$
$\log_e 2 = 0.6931$
$\log_e 10 = 2.3026, \log_{10} e = 0.4343$

(*b*) $\log_a p + \log_a q = \log_a pq$

(*c*) $\log_a p - \log_a q = \log_a \dfrac{p}{q}$

(*d*) $\log_a p^q = q \log_a p$

7. Nature of Trigonometric Ratios in Quadrant



8. Product and Sum Formulae for trigonometric functions

(*a*) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(*b*) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(*c*) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(*d*) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(*e*) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(*f*) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(*g*) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(*h*) $\cos 2A = \cos^2 A - \sin^2 A$
$= 1 - 2\sin^2 A$
$= 2\cos^2 A - 1 = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(*i*) $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{2 \tan A}{1 - \tan^2 A}$

(*j*) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(*k*) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(*l*) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

(*m*) $\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

(*n*) $\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

(*o*) $\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

(*p*) $\cos A - \cos B = 2 \sin \dfrac{A+B}{2} \sin \dfrac{B-A}{2}$

(*q*) $\sin A \cos B = \dfrac{1}{2} [\sin(A+B) + \sin(A-B)]$

(*r*) $\cos A \sin B = \dfrac{1}{2} [\sin(A+B) - \sin(A-B)]$

(*s*) $\cos A \cos B = \dfrac{1}{2} [\cos(A+B) + \cos(A-B)]$

(*t*) $\sin A \sin B = \dfrac{1}{2} [\cos(A-B) - \cos(A+B)]$

(*u*) $\sin x = 0 \Leftrightarrow x = n\pi, n \in Z$

(*v*) $\sin x = \pm 1 \Leftrightarrow x = (4n \pm 1) \dfrac{\pi}{2}, n \in Z$

(*w*) $\cos x = 0 \Leftrightarrow x = (2n + 1) \dfrac{\pi}{2}, n \in Z$

(*x*) $\cos x = \pm 1 \Leftrightarrow x = 2n\pi$ and $x = (2n + 1)\pi$, $n \in Z$

(*y*) $e^{ax} \neq 0, \forall x \in R; a \in R$

9. Basic differentiation formulae

(*a*) $\dfrac{d}{dx} (\sin x) = \cos x$

(*b*) $\dfrac{d}{dx} (\cos x) = -\sin x$

(*c*) $\dfrac{d}{dx} (\tan x) = \sec^2 x$

(d) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(e) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(f) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(g) $\frac{d}{dx}(e^x) = e^x$

(h) $\frac{d}{dx}(a^x) = a^x \log_e a$

(i) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(j) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(k) $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

(l) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$

(m) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$

(n) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(o) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(p) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, x \neq 0, \pm 1$

(q) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}, x \neq 0, \pm 1$

(r) $\frac{d}{dx}(\sin hx) = \cos hx$

(s) $\frac{d}{dx}(\cos hx) = -\sin hx$

10. Basic Integration Formulae

(a) $\int \sin x \, dx = -\cos x + c$

(b) $\int \cos x \, dx = \sin x + c$

(c) $\int \tan x \, dx = -\log \cos x + c = \log \sec x + c$

(d) $\int \cot x \, dx = \log \sin x + c$

(e) $\int \sec x \, dx = \log(\sec x + \tan x) + c$

(f) $\int \text{cosec}\, x \, dx = \log(\text{cosec}\, x - \cot x) + c$

(g) $\int \sec^2 x \, dx = \tan x + c$

(h) $\int \text{cosec}^2 x \, dx = -\cot x + c$

(i) $\int e^x dx = e^x$

(j) $\int a^x dx = \frac{a^x}{\log_e a} + c; a > 0, a \neq 1$

(k) $\int \frac{1}{x} dx = \log_e x + c$

(l) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

(m) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(n) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$

(o) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

(p) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$

(q) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sin h^{-1} \frac{x}{a} + c$

(r) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cos h^{-1} \frac{x}{a} + c$

(s) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

(t) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

ABBREVIATIONS

lim	–	limit
$\therefore$	–	therefore
$\because$	–	because of
i.e.,	–	that is
$f^n(a)$	–	*n*th derivative of (f) at 'a'
sup.	–	supremum
Inf.	–	infimum
$Lf'(a)$	–	left hand derivative of 'f' at 'a'
$Rf'(a)$	–	right hand derivative of 'f' at 'a'
$Lf(a)$	–	left hand limit of 'f' at 'a'
$Rf(a)$	–	right hand limit of 'f' at 'a'
diag.	–	diagonal
L.H.S.	–	left hand side
R.H.S.	–	right hand side
dim	–	dimension
adj (A)	–	adjoint of matrix A
min.	–	minimum
max.	–	maximum
L.C.	–	linear combination
L.D.	–	linear dependence
L.I.	–	linear independence

List of Figures

Unit 1: Calculus I

Unit 2: Calculus II

Unit 3: Sequence and Series

Unit 4: Multivariable Calculus

Unit 5: Matrices

Guidelines for Teachers

To implement Outcome Based Education (OBE) knowledge level and skill set of the students should be enhanced. Teachers should take a major responsibility for the proper implementation of OBE. Some of the responsibilities (not limited to) for the teachers in OBE system may be as follows:

- Within reasonable constraint, they should manipulate time to the best advantage of all students.
- They should assess the students only upon certain defined criterion without considering any other potential ineligibility to discriminate them.
- They should try to grow the learning abilities of the students to a certain level before they leave the institute.
- They should try to ensure that all the students are equipped with the quality knowledge as well as competence after they finish their education.
- They should always encourage the students to develop their ultimate performance capabilities.
- They should facilitate and encourage group work and team work to consolidate newer approach.
- They should follow Blooms taxonomy in every part of the assessment.

Bloom's Taxonomy

Level	Teacher should Check	Student should be able to	Possible Mode of Assessment
Creating	Students ability to create	Design or Create	Mini project
Evaluating	Students ability to Justify	Argue or Defend	Assignment
Analysing	Students ability to distinguish	Differentiate or Distinguish	Project/Lab Methodology
Applying	Students ability to use information	Operate or Demonstrate	Technical Presentation/ Demonstration
Understanding	Students ability to explain the ideas	Explain or Classify	Presentation/Seminar
Remembering	Students ability to recall (or remember)	Define or Recall	Quiz

Guidelines for Students

Students should take equal responsibility for implementing the OBE. Some of the responsibilities (not limited to) for the students in OBE system are as follows:

- Students should be well aware of each UO before the start of a unit in each and every course.
- Students should be well aware of each CO before the start of the course.
- Students should be well aware of each PO before the start of the programme.
- Students should think critically and reasonably with proper reflection and action.
- Learning of the students should be connected and integrated with practical and real life consequences.
- Students should be well aware of their competency at every level of OBE.

Contents

1 Calculus I

UNIT SPECIFICS

This unit elaborately discusses about the topics curvature, radius of curvature, centre of curvature, circle of curvature, evolutes, involute, envelope, definite and improper integrals, beta and gamma functions and their properties, applications of definite integrals to evaluate surface areas and volumes of revolutions. All the above topics have been discussed with ample examples so as to make theory application crystal clear to the students. Figures also included wherever required to make students visualize the topics.

RATIONALE

Involute and Evolute is a part of Differential geometry,which is itself a very important concept for students who are working in the field of Science, Artificial Intelligence and Robotics. It has many applications in day-to-day real life also.

One of the major application of Involute of circle is in designing of gears for revolving parts where gear tooth follow the shape of involute.

The basic application of involute usage is in winding clocks and toys where a winding key is used to motion the spiral spring in a circular involute.

We use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Indefinite integrals are used to find areas and volumes of curves of bounded bodies.

Gamma function is used in gamma distribution which is used to determine time based occurrence, such as life span of an electronic component.

PRE-REQUISITES

1. Basic knowledge of integration and differentiation.
2. Understanding of different curves like circle, ellipse, hyperbola etc.
3. Familiar with the concept of factorial.
4. Use integration to find out the area enclosed by two or more curves.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U1-01: Explain the concept of curvature and radius of curvature; also find the evolutes of curve with the help of centre of curvature.

U1-02: Apply integral test on various functions to find their nature in terms of convergence and divergence.

U1-03: Familiarise themselves with the concept of Beta-Gamma functions and apply these to evaluate various integrals.

U1-04: Evaluate the surface area and the volume of solids of revolution for cartesian, parametric and polar curves.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 1 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U1-01	3	1	–	–	1
U1-02	2	3	–	–	–
U1-03	3	–	–	–	–
U1-04	2	–	–	–	1

HISTORY

Apollonius (c. 262–190 BC) "calculated" curvature of conic sections implicitly when solving the problem of drawing normals to them in book V of Conica, but he did not think of it as a property of a curve, and his "calculations" are constructions of segments. The first person to "see" curvature was Oresme (c. 1320-1382), Descartes's precursor in introducing coordinates. He described it as a local measure of curve's bending, and christened it with the Latin "curvitas". Later he proposed that for circles it can be quantified by the reciprocal of the radius, our modern convention. Kepler vaguely suggested how to define curvature for general curves by considering the "closest" circle at a point, named osculating circle by Leibniz in 1680s. But it was Huygens, who first found a way to calculate curvature for general curves, and Newton who gave the concept its modern form.

"I believe that we do not know anything for certain everything probably."

—Christiaan Huygens

1.1 CURVATURE

In the Fig. 1.1, it can be seen that the given curve *AMNB* bends more sharply at the point *M* as comparison to the point *N*. The bending of a curve at a particular point is called the curvature of the curve at that point. So the curvature at *M* is more than the curvature at *N*. It will give a definite numerical measure of the sharpness of the bending of the curve at the point.

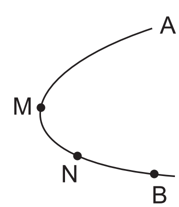


Fig. 1.1

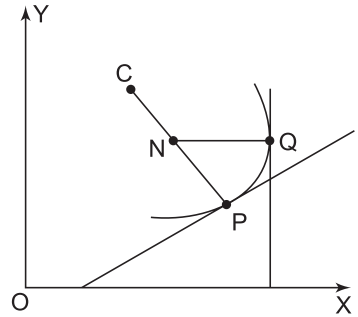


Fig. 1.2

In Fig. 1.2, let P be any point on a given curve and Q be a neighbouring point of P such that the arc PQ is concave towards its chord. Let the normals at P and Q intersects at N.

When $Q \to P$, N tends to a definite position C, called the centre of curvature of the curve at P. The distance CP is called the radius of curvature of the curve at the point P and is denoted be ρ (rho). The circle with centre at C and the radius B, is equal to CP, is called the circle of curvature of the given curve at the point P. Any chord of the circle at curvature drawn through the point P is called the chord of curvature. The reciprocal of the radius of curvature is called the curvature of the curve at the point P and is denoted by κ.

1.1.1 Mathematical Definition of a Curvature

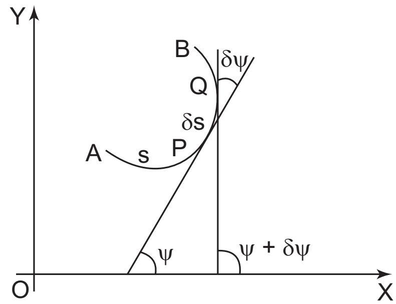


Fig. 1.3

Let AB be a curve and P, Q be two neighbouring points on this curve. Let an arc $AP = s$ and the arc $AQ = s + \delta s$. 'A' is a fixed point on the curve from which arcs length are measured. Let the tangents at P and Q makes an angle ψ and $\psi + \delta\psi$ respectively with a fixed line, *i.e.*, x-axis, then

i. The angle $\delta\psi$ through which the tangent turns as its point of the contact travels along the arc PQ is called the total bending or total curvature of the arc PQ.

ii. The ratio $\dfrac{\delta\psi}{\delta s}$ is called the mean or average curvature of the arc PQ.

iii. The limiting value of the mean curvature which $Q \to P$ is called the curvature of the curve at the point P.

Thus, the curvature (κ) at point P is $\underset{Q\to P}{\text{Lt}} \dfrac{\delta\psi}{\delta s} = \underset{\delta s\to 0}{\text{Lt}} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds}$.

iv. The reciprocal of the curvature of the curve at P, provided this curvature is not zero, is called the radius of curvature of the curve at P and is denoted by ρ *i.e.* $\rho = \dfrac{1}{\kappa} = \dfrac{ds}{d\psi}$.

Remarks: (1) A straight line does not bend at all (as ψ is constant, so $\dfrac{d\psi}{ds}$ is zero).

Hence curvature of a straight line is zero.

(2) Curvature of a circle is constant and equal to the reciprocal of its radius.

1.1.2 Radius of Curvature

The reciprocal of the curvature at any point is called the radius of curvature at that point. Obviously the curvature at any point should not be zero for defining radius of curvature at that point. It is usually denoted by ρ. Hence

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi} \text{ at } P.$$

Graphically

For the curve CD, if ρ is the radius of curvature at point P, we draw normal at the point P and then $O'P$ is the distance equal to radius of curvature at P with O'.

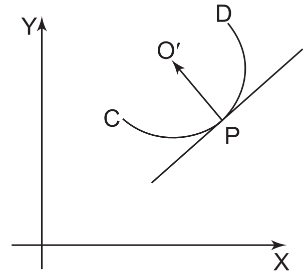


Fig. 1.4

A. Radius of Curvature for the Cartesian Curve

We find the expression for ρ, when the equation of curve is given in cartesian co-ordinates.

To find ρ for the curve $y = f(x)$

If ψ is the angle which the tangent at $P(x, y)$ on the curve makes with x-axis, then we have

$$\sin\psi = \frac{dy}{ds},\ \cos\psi = \frac{dx}{ds} \text{ and } \tan\psi = \frac{dy}{dx}$$

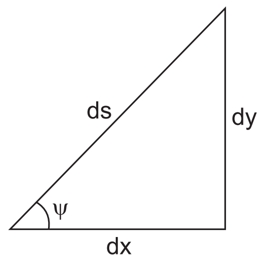


Fig. 1.5

So from the last relation, we have

$$\psi = \tan^{-1}(y_1) \text{ where } y_1 = \frac{dy}{dx}$$

Differentiating w.r.t x, we have

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\cdot y_2 \qquad \left\{\because\ y_2 = \frac{d^2y}{dx^2}\right\}$$

Now,
$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\psi}{dx}\cdot\frac{dx}{ds} = \frac{y_2}{1+y_1^2}\cdot\cos\psi$$

$$= \frac{y_2}{1+y_1^2}\cdot\frac{1}{\sqrt{1+y_1^2}}$$

$$\left[\because\ \cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}} = \frac{1}{\sqrt{1+(dy/dx)^2}} = \frac{1}{\sqrt{1+y_1^2}}\right]$$

So,
$$\frac{1}{\rho} = \frac{y_2}{(1+y_1^2)^{3/2}}$$

$$\rho = \frac{[1+y_1^2]^{3/2}}{y_2} \text{ where } y_2 \neq 0$$

B. Radius of Curvature for Parametric Curve

To find ρ for the curve $x = f(t), y = \phi(t)$ *i.e.*, when the parametric equation of the curve is given.

Here $x = f(t)$ and $y = \phi(t)$, t being parameter

We know,
$$x' = \frac{dx}{dt},\ y' = \frac{dy}{dt}$$

Also, $$x'' = \frac{d^2x}{dt^2}, y'' = \frac{d^2y}{dt^2}$$

Thus, $$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} \quad ...(1)$$

Differentiating (1) w.r.t. x, we have

$$y_2 = \frac{x'y'' - y'x''}{x'^2} \cdot \frac{dt}{dx} = \frac{x'y'' - y'x''}{x'^3} \qquad \left[\because \frac{dt}{dx} = \frac{1}{x'}\right]$$

We know, $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$ so, after putting all values, we have

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

C. Radius of Curvature for Polar Curve

To find ρ for the curve $r = f(\theta)$ or $f(r, \theta) = 0$

Let ϕ be the angle which the tangent at $P(r, \theta)$ makes with OP, then we have

$$\tan\phi = \frac{rd\theta}{dr}, \quad \sin\phi = \frac{rd\theta}{ds} \quad \text{and} \quad \cos\phi = \frac{dr}{ds} \quad ...(1)$$

Again if ψ be the angle which the tangent $P(r, \theta)$ makes with OX. then

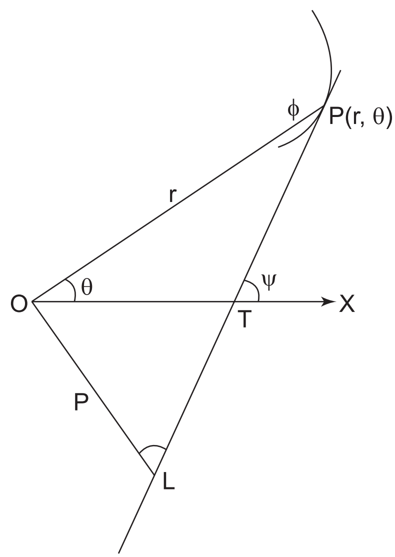


Fig. 1.6

$$\psi = \theta + \phi$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left[1 + \frac{d\phi}{d\theta}\right] \quad ...(2)$$

From (1), $$\tan\phi = r \cdot \frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

or $$\tan\phi = \frac{r}{r_1} \quad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating both side w.r.t θ, we have

$$\sec^2\phi \cdot \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - rr_2}{r_1^2}$$

$$\Rightarrow \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{1}{\sec^2\phi}$$

$$= \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{1}{1+\tan^2\phi}$$

$$= \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{r_1^2}{r_1^2 + r^2} \qquad \left[\because \tan\phi = \frac{r}{r_1}\right]$$

$$= \frac{r_1^2 - rr_2}{r_1^2 + r^2} \qquad ...(3)$$

Again from (1) $\qquad r\frac{d\theta}{ds} = \sin\phi = \frac{1}{\operatorname{cosec}\phi}$

$$\Rightarrow \qquad \frac{d\theta}{ds} = \frac{1}{r} \cdot \frac{1}{\sqrt{1+\cot^2\phi}} = \frac{1}{\sqrt{r^2 + r_1^2}} \qquad ...(4)$$

From (2), (3) and (4), we have

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{r^2 + r_1^2}} \cdot \left[1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2}\right]$$

$$\frac{1}{\rho} = \frac{2r_1^2 + r^2 - rr_2}{(r^2 + r_1^2)^{3/2}} \qquad \left\{\because \frac{1}{\rho} = \frac{d\psi}{ds}\right\}$$

$$\therefore \qquad \rho = \frac{ds}{d\psi} = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}$$

SOME SOLVED EXAMPLES

Example 1.1. *Find the radius of curvature at the given point of following curves:*

a. $y = 4\sin x - \sin 2x$ *at* $x = \frac{\pi}{2}$ $\qquad$ b. $\sqrt{x} + \sqrt{y} = 1$ *at the point* $\left(\frac{1}{4}, \frac{1}{4}\right)$

Solution. (*a*) Given, $\qquad y = 4\sin x - \sin 2x \qquad ...(1)$

Differentiate (1) w.r.t x

$$\frac{dy}{dx} = 4\cos x - 2\cos 2x$$

and $\qquad \frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$

We know, $\qquad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + (4\cos x - 2\cos 2x)^2\right]^{3/2}}{-4\sin x + 4\sin 2x}$

$$\therefore \qquad \rho \text{ at } \frac{\pi}{2} = \frac{\left[1+\left(4\cos\frac{\pi}{2}-2\cos\pi\right)^2\right]^{3/2}}{-4\sin\frac{\pi}{2}} = \frac{(1+4)^{3/2}}{-4} \qquad \text{(ignoring the negative sign)}$$

$$= \frac{5^{3/2}}{4} \qquad \textbf{(Answer)}$$

b. Given, $\sqrt{x}+\sqrt{y} = 1$ ...(1)

Differentiating the equation (1) w.r.t x

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{y}}} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$= -1 \text{ at } \left(\frac{1}{4},\frac{1}{4}\right)$$

Similarly, we have
$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{x}\,.\,\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\,.\,\frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{\left[\frac{\sqrt{x}}{2\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)-\frac{\sqrt{y}}{2\sqrt{x}}\right]}{x} = \frac{\left[-\frac{1}{2}-\frac{\sqrt{y}}{2\sqrt{x}}\right]}{x}$$

$$= -\left[\frac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}\right] = \frac{-\left(\frac{1}{2}+\frac{1}{2}\right)}{\left(2\,.\,\frac{1}{4}\,.\,\frac{1}{2}\right)} \qquad \left(\text{at } \frac{1}{4},\frac{1}{4}\right)$$

$$= -4$$

We know,
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{-(1+1)^{3/2}}{4} = \frac{2^{3/2}}{4} = \frac{1}{\sqrt{2}} \qquad \textbf{(Answer)}$$

Example 1.2. *Find the least value of* $|\rho|$ *for* $y = \log x,\ x > 0$.

Solution. Let $y = \log x$ ...(1)

Differentiating (1) w.r.t x, we have

$$\frac{dy}{dx} = \frac{1}{x} \text{ and } \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

We know, $$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2} = \frac{\left(1+\frac{1}{x^2}\right)^{3/2}}{-\frac{1}{x^2}}$$

$$= \frac{-(x^2+1)^{3/2}}{x}$$

Let $$|\rho| = f(x) = \frac{(x^2+1)^{3/2}}{x}$$

To find the least value of $f(x)$, we find $f'(x)$

$$f'(x) = \frac{x\,.\,\frac{3}{2}(x^2+1)^{1/2}\,2x - (x^2+1)^{3/2}}{x^2}$$

$$= \frac{3x^2\sqrt{x^2+1} - (x^2+1)^{3/2}}{x^2}$$

$$= \frac{\sqrt{x^2+1}\,(3x^2-x^2-1)}{x^2} = \frac{\sqrt{x^2+1}\,(2x^2-1)}{x^2}$$

Equate $f'(x) = 0$, we get $x = \pm\frac{1}{\sqrt{2}}$

and also $f''(x)$ is positive for $x = \frac{1}{\sqrt{2}}$ (students can check)

Hence $|\rho|$ is minimum for $x = \frac{1}{\sqrt{2}}$ $(\because \quad x > 0)$

$$\therefore \qquad |\rho|_{\min} = \left[\frac{(x^2+1)^{3/2}}{x}\right]_{x=\frac{1}{\sqrt{2}}} = \frac{\left(\frac{1}{2}+1\right)^{3/2}}{\frac{1}{\sqrt{2}}}$$

$$= \left(\frac{3}{2}\right)^{3/2}\sqrt{2} = \frac{3\sqrt{3}}{2}$$

$\therefore$ Minimum value of $|\rho|$ is $\frac{3\sqrt{3}}{2}$. **Answer**

Example 1.3. *Find the radius of curvature for Rectangular hyperbola $xy = c^2$ at the point (x, y).*

Solution. Hint: Take $y = \frac{c^2}{x}$. **Answer:** $\frac{(x^2+y^2)^{3/2}}{2c^2}$.

Example 1.4. *Find the radius of curvature at the origin of the two branches of the curve given by $x = 1 - t^2$, $y = t - t^3$.*

Solution. At origin (0, 0) two common value for t are 1 and –1. $[\because \quad x = 1 - t^2, 0 = 1 - t^2$ at $x = 0$

Hence for two branches of the curves, value of t are 1 and –1 $\quad 1 = t^2$

$t = \pm 1]$

Given, $x = 1 - t^2$, $y = t - t^3$

Differentiating the above w.r.t 't', we get

$$\frac{dy}{dt} = 1 - 3t^2, \quad \frac{dx}{dt} = -2t$$

$$\frac{dy}{dx} = \frac{1-3t^2}{-2t} = -\frac{1}{2t} + \frac{3}{2}t \qquad \text{...(1)}$$

Again differentiating (1) w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2t^2} + \frac{3}{2}\right) \cdot \frac{dt}{dx} = \frac{\left(\frac{1}{2t^2} + \frac{3}{2}\right)}{-2t}$$

$$= \frac{-1}{4t^3} - \frac{3}{4t}$$

We find, $$\left(\frac{dy}{dx}\right)_{t=1} = -\frac{1}{2} + \frac{3}{2} = 1$$

and $$\left(\frac{d^2y}{dx^2}\right)_{t=1} = -\frac{1}{4} - \frac{3}{4} = -1$$

Similarly, $$\left(\frac{dy}{dx}\right)_{t=-1} = \frac{1}{2} - \frac{3}{2} = -1$$

and $$\left(\frac{d^2y}{dx^2}\right)_{t=-1} = \frac{1}{4} + \frac{3}{4} = 1$$

So, $$(\rho)_{t=1} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{3/2}}{-1} = -2\sqrt{2}$$

and $$(\rho)_{t=-1} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2} \quad \textbf{(Answer)}$$

Example 1.5. *Prove that the radius of curvature for the catenary $y = c \cos h \dfrac{x}{c}$ is equal to the portion of the normal intercepted between the curve and the x-axis and that it varies as the square of the ordinate.*

Solution. Try yourself.

Example 1.6. *If ρ_1 and ρ_2 be the radii of curvature at the ends of focal chord of the parabola $y^2 = 4ax$, then prove that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.*

Solution. The equation of parabola is $y^2 = 4ax$, which in parametric form is $x = at^2$, $y = 2at$. If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the two extremities of a focal chord of the parabola, then

$$t_1 t_2 = -1$$

For $\quad x = at^2,\ y = 2at$

We have, $$\frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{t}$$

and $$\frac{d^2y}{dx^2} = \frac{-1}{t^2}\cdot\frac{dt}{dx} = -\frac{1}{2at^3}$$

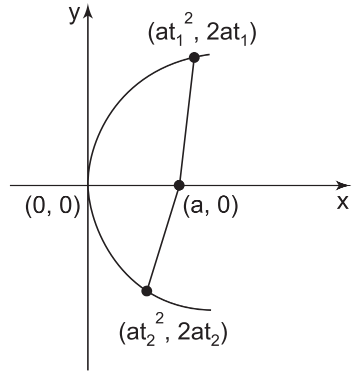


Fig. 1.7

We know, $$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{1}{t^2}\right]^{3/2}}{-\frac{1}{2at^3}}$$

$$= -2a\ (t^2+1)^{3/2}$$

$\therefore$ $$\rho \text{ at } (at_1^2, 2at_1) = -2a\ (t_1^2+1)^{3/2}$$

and $$\rho \text{ at } (at_2^2, 2at_2) = -2a\,(t_2^2+1)^{3/2}$$

$\therefore$ $$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = \left[2a\,(t_1^2+1)^{3/2}\right]^{-2/3} + \left[2a\,(t_2^2+1)^{3/2}\right]^{-2/3}$$

$$= (2a)^{-2/3}\left[\frac{1}{t_1^2+1} + \frac{1}{t_2^2+1}\right] = (2a)^{-2/3}\left[\frac{t_2^2+1+t_1^2+1}{(t_1^2+1)(t_2^2+1)}\right]$$

$$= (2a)^{-2/3}\left[\frac{t_1^2+t_2^2+2}{t_1^2+t_2^2+(-1)^2+1}\right] = (2a)^{-2/3} \quad \textbf{Proved.} \qquad [\because\ t_1t_2 = -1]$$

Example 1.7. *Show that the radius of curvature at the end of the major axis of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is equal to the semi-latus rectum of the ellipse.*

Solution. Equation of an ellipse in parametric form is $x = a \cos t$, $y = b \sin t$

and the ends of major axis are $(\pm\, a, 0)$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = -\,a \sin t,\ \frac{dy}{dt} = b \cos t$$

$\therefore$ $$\frac{dy}{dx} = \frac{-b}{a} \cot t$$

and $$\frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 t\ \frac{dt}{dx} = \frac{b}{a} \operatorname{cosec}^2 t \times \frac{1}{-a \sin t}$$

$$= \frac{-b}{a^2} \operatorname{cosec}^3 t$$

Radius of curvature is, $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \dfrac{\left[1+\dfrac{b^2\cos^2 t}{a^2\sin^2 t}\right]^{3/2}}{\dfrac{-b}{a^2}\operatorname{cosec}^3 t}$

$= \dfrac{1}{ab}(a^2\sin^2 t + b^2\cos^2 t)^{3/2}$ (ignoring the negative sign)

$= \rho$ at $(a, 0)$ is $\dfrac{1}{ab}(a^2\sin^2 0 + b^2\cos^2 0)^{3/2}$

$= \dfrac{b^2}{a}$ (Semi-latus rectum of the ellipse) **Proved.**

Example 1.8. *Find the radius of curvature for the curve* $x = c \log\,(s+\sqrt{s^2+c^2}),\ y=\sqrt{s^2+c^2}$

Solution. Given, $x = c\log\,(s+\sqrt{s^2+c^2})$

Differentiating w.r.t. s, we get

$$\frac{dx}{ds} = \frac{c}{s+\sqrt{s^2+c^2}}\left(1+\frac{2s}{2\sqrt{s^2+c^2}}\right) = \frac{c}{\sqrt{s^2+c^2}}$$

and $$\frac{dy}{ds} = \frac{2s}{2\sqrt{s^2+c^2}} = \frac{s}{\sqrt{s^2+c^2}}$$

So, $$\frac{dy}{dx} = \frac{\dfrac{dy}{ds}}{\dfrac{dx}{ds}} = \frac{s}{c} \qquad \text{...(1)}$$

We know, $\tan\psi = \dfrac{dy}{dx}$

So, $s = c\tan\psi$ [from (1)]

$\therefore$ Radius of curvature, $\dfrac{ds}{d\psi} = c\sec^2\psi$

$= c\,(1+\tan^2\psi)$

$= c\left(1+\dfrac{s^2}{c^2}\right)$

$= \dfrac{c^2+s^2}{c}$ **(Answer)**

Example 1.9. *Prove that in the curve* $r^2 = a^2 \sin 2\theta$*, the curvature varies as the radius vector.*

Solution. Try yourself.

Example 1.10. *If* ρ_1 *and* ρ_2 *are the radii of curvature at the extremities of any chord through the pole of the cardioid* $r = a\,(1 - \cos\theta)$*, show that* $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$.

Solution. The cardioid $r = a(1 - \cos\theta)$ is as shown in Fig. 1.8.

If the point P_1 is (r_1, θ), then P_2 would be $(r_2, \theta + \pi)$ as P_1 and P_2 are the extremities of the chord through pole.

then, $$\frac{dr}{d\theta} = r_1 = a \sin\theta$$

and $$\frac{d^2r}{d\theta^2} = r_2 = a\cos\theta$$

then, $$\rho = \frac{(r^2 + r_1^{\,2})^{3/2}}{r^2 + 2r_1^{\,2} - rr_2}$$

$$= \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1-\cos\theta)\cos\theta}$$

$$= \frac{a^3(1\cos\theta)^{3/2}\, 2\sqrt{2}}{3a^2\,(1-\cos\theta)} = \frac{a}{3} 2\sqrt{2}\,(1-\cos\theta)^{1/2}$$

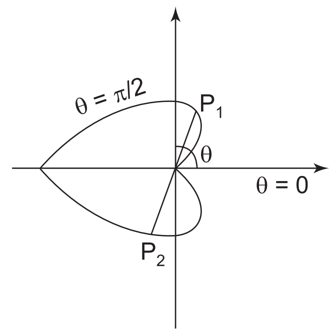


Fig. 1.8

$\therefore$ $$\rho_1 = \frac{2\sqrt{2}}{3} a(1-\cos\theta)^{1/2}$$

and, $$\rho_2 = \frac{2\sqrt{2}}{3} a[1-\cos(\theta+\pi)]^{1/2}$$

$$= \frac{2\sqrt{2}}{3} a(1+\cos\theta)^{1/2}$$

$\therefore$ $$\rho_1^{\,2} + \rho_2^{\,2} = \frac{8a^2}{9}(1 - \cos\theta + 1 + \cos\theta) = \frac{16a^2}{9}$$

$$\rho_1^{\,2} + \rho_2^{\,2} = \frac{16a^2}{9}.$$ **Proved.**

EXERCISE 1.1

Find the radius of curvature at the given point of the following curves:

1. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

2. $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

3. $y^2 = \frac{a^2\,(a-x)}{x}$ at $(a, 0)$ (**Hint:** Equation of curvature is $x = \frac{a^3}{y^2 + a^2}$)

4. $x^2y = a(x^2 + y^2)$ at $(-2a, 2a)$

5. Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.

6. Find the radius of curvature at any point $(0, c)$ of the catenary $y = c \cos h\, \frac{x}{c}$.

7. Show that for the parabola $y^2 = 4ax$, ρ^2 varies as $(SP)^3$, where ρ is the radius of curvature at any point P of the parabola and S is the focus of the parabola.

8. The tangent at two points P, Q on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angles: show that if ρ_1, ρ_2 be the radii of curvature at these points, then $\rho_1^2 + \rho_2^2 = 16a^2$.

9. Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the length of the perpendicular from the origin to the tangent at the point.

10. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \rho = \frac{a^2b^2}{P^3}$ where P is the perpendicular from the center upon the tangent at (x, y).

11. Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$.

12. Show that for the curve; $x = a \cos \theta\ (1 + \sin \theta)$, $y = a \sin \theta\ (1 + \cos \theta)$, the radius of curvature at $\theta = \frac{-\pi}{4}$ is a.

Find the radius of curvature for the following curves:

13. $r = a \cos n\theta$

14. $r^m = a^m \sin m\theta$

15. $r^2 \cos 2\theta = a^2$

16. Show that radius of curvature at any point of the curve $r = a \cos n\theta$, where $r = a$ is $\frac{a}{1+n^2}$.

17. Show that radius of curvature for the curve $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}$ is $r\sqrt{r^2 - a^2}$.

18. Show that the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\theta$ at the point where the tangent is parallel to x-axis is $\frac{\sqrt{2}}{3}a$.

Answers

1. $\frac{a}{\sqrt{2}}$

2. $\frac{3a}{8\sqrt{2}}$

3. $\frac{a}{2}$

4. 2

5. $2\sqrt{2}$

6. c

11. $\left(\frac{9}{8}, 3\right)$ and $\left(\frac{9}{8}, -3\right)$

13. $\frac{(r^2 + a^2n^2 - r^2n^2)^{3/2}}{r^2 - r^2n^2 + 2a^2n^2}$

14. $\frac{a^m}{(m+1)^{r^{m-1}}}$

15. $\frac{r^3}{a^2}$

1.1.3 Centre of Curvature, Circle of Curvature

1.1.3.1 Centre of Curvature

The centre of curvature at any point P of a curve AB is the point which lies on the positive direction of the normal at P and is at a distance equal to the radius of curvature from it.

1.1.3.2 Circle of Curvature

If 'C' is the centre of curvature, then the circle with centre 'C' and radius of curvature 'ρ' passing through the point P, is called the circle of curvature.

Let ρ be the radius of curvature and $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature at a given point, then the equation of the circle of curvature is given by $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

1.1.4 Coordinates of the Centre of Curvature

Let $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature C, lying on the normal at $P(x, y)$ on the curve such that $PC = \rho$.

From the figure 1.9, we have

$$\bar{x} = OL = OM - LM$$

$$= OM - PQ \qquad (\because \quad LM = PQ)$$

Now $\quad OM = x$

$$PQ = PC \sin\psi = \rho \sin\psi$$

$\therefore$
$$\bar{x} = x - \rho\sin\psi \qquad \text{...(1)}$$

Similarly,
$$\bar{y} = CL = CQ + QL$$

$$= CQ + PM \qquad (\because \quad QL = PM)$$

Now $\quad CQ = PC\cos\psi = \rho\cos\psi$ and $PM = y$

$\therefore$
$$\bar{y} = y + \rho\cos\psi \qquad \text{...(2)}$$

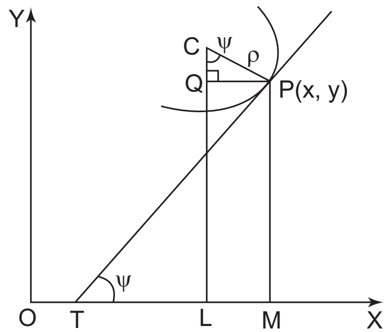


Fig. 1.9

As
$$\tan\psi = \frac{dy}{dx} = y_1$$

$\therefore$
$$\sin\psi = \tan\psi \,.\, \cos\psi$$

$$= \frac{\tan\psi}{\sec\psi} = \frac{\tan\psi}{\sqrt{1+\tan^2\psi}} \quad \text{or} \quad \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}$$

Using the above values of $\sin\psi$ and $\cos\psi$ in equations (1) and (2), we get

$$\bar{x} = x - \rho\frac{y_1}{\sqrt{1+y_1^2}} \quad \text{and} \quad \bar{y} = y + \frac{\rho}{\sqrt{1+y_1^2}}$$

But, as we know
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

So, we have,
$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2) \qquad \text{...(3)}$$

and
$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2) \qquad \text{...(4)}$$

are the required coordinates of the centre of curvature.

If we eliminate x, y between the equations (3) and (4) and the equation of the curve, we obtain a relation between $\bar{x}$ and $\bar{y}$ which is the equation of the evolute.

HISTORY

Huygens named the locus of the centers of curvature to a curve its evolute, and showed how to construct a perfect pendulum, whose period does not depend on its amplitude. The construction was based on the fact that evolute to a cycloid is congruent to it.

1.1.5 Evolute

Corresponding to each point on a curve we can find the curvature of the curve at that point. Drawing the normal at these points, we can find Centre of Curvature corresponding to each of these points. Since the curvature varies from point to point, centre's of curvature also differ. The totality of all such centres of curvature of a given curve will define another curve and this curve is called the evolute of the curve.

The Locus of centres of curvature of a given curve is called the evolute of that curve. The locus of the centre of curvature C of a variable point P on a curve is called the evolute of the curve. The curve itself is called involute of the evolute.

Here, for different points on the curve, we get different centre of curvatures. The locus of all these centres of curvature is called as Evolute. The external curve which satisfies all these centres of curvature is called as Evolute. Here Evolute is nothing but an curve equation.

To find Evolute, the following models exist.

If an equation of the curve is given and if we have to prove, L.H.S = R.H.S., then following steps should be followed:

1. First find Centre of Curvature, $C(\bar{x}, \bar{y})$ where $\bar{x} = x - [y_1(1 + y_1^2)]/y_2$, $\bar{y} = y + [(1 + y_1^2)]/y_2$, and then consider L.H.S: In that directly substitute $\bar{x}$ in place of x and $\bar{y}$ in place of y. Similarly for R.H.S. and then show that L.H.S = R.H.S.
2. If a curve is given and if we are asked to find the evolute of the given curve, then do as follows: First find Centre of curvature $C(\bar{x}, \bar{y})$ and then re-write as x in terms of $\bar{x}$ and y in terms of $\bar{y}$ and then substitute in the given curve, which gives us the required evolute.
3. If a curve is given, which is in parametric form, then first find Centre of curvature, which will be in terms of parameter. Then using these values of $\bar{x}$ and $\bar{y}$ eliminate the parameter, which gives us evolute.

Pictorial representation of Evolute and Centre of Curvature in Fig. 1.10.

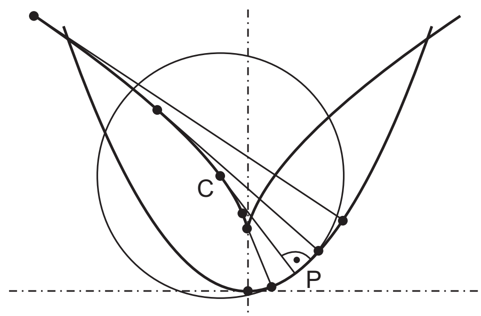


Fig. 1.10

1.1.6 Involute

A curve that is obtained by attaching a string which is imaginary and then winding and unwinding it tightly on the curve given is called involute in differential geometry. Involute or evolvent is the locus of the free end of this string.

For more Clarifications: The **evolute** of an involute of a curve is referred to that original curve. In other words, the locus of the center of curvature of a curve is called evolute and the traced curve itself is known as the involute of its evolute.

Remark: This is a part of a special branch of geometry called differential Geometry of Curves. It talks about the smooth curves which lie in Euclidean space and has applications of different methods of integral and differential calculus on them. The shapes related to some other curves are called involutes. This was discovered by Christine Huygens in 1673. He was a Dutch mathematician and a physicist.

1.1.6.1 Involutes of the Curves

Here we will see the involutes of the different curves as shown below:

- Involute of a Circle
- Involute of a Catenary
- Involute of a Deltoid
- Involute of a Parabola
- Involute of an Ellipse

1. **Involute of a Circle:** It is similar to the Archimedes spiral.

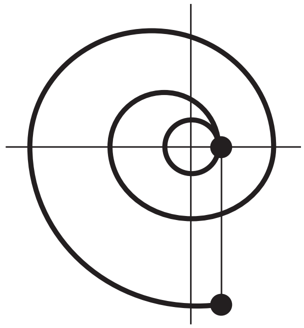


Fig. 1.11. Involute of a Circle

2. **Involute of a Catenary:** It is a curve which is similar to hanging cable supported by its ends. So, it is a *U* shaped hanging chain which looks like a parabola. The tractrix is the involute of the catenary through the vertex.

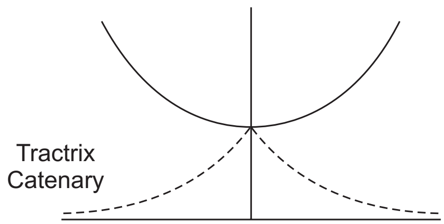


Fig. 1.12. Involute of a Catenary

3. **Involute of a Deltoid:** It is a tricuspid curve with three cusps. It resembles greek letter delta.

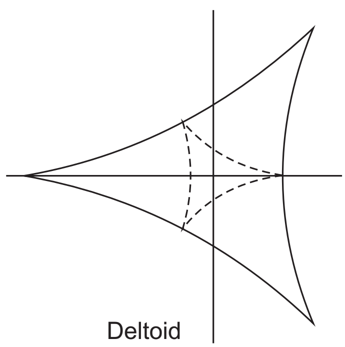


Fig. 1.13. Involute of a Deltoid

4. **Involute of a Parabola:**

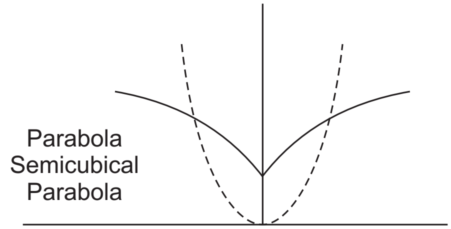


Fig. 1.14. Involute of a Parabola

5. **Involute of an Ellipse:**

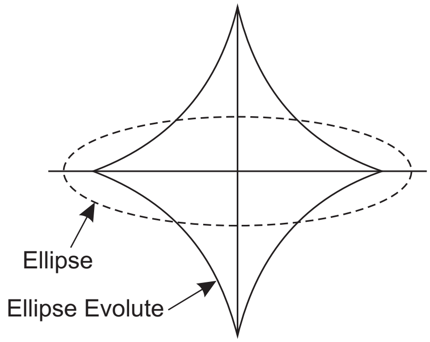


Fig. 1.15. Involute of an Ellipse

The following equations are used for defining the given:

- Circle Involute
- Catenary Involute
- Deltoid Involute
- Parabola Involute
- Parabola Involute

Circle Involute: $x = r(\cos t + t \sin t)$, $y = r(\sin t - t \cos t)$, where, r = radius of the circle, t = parameter of angle in radian.

Catenary Involute: $x = t - \tanh t$, $y = \operatorname{sech} t$, where t be the parameter.

Deltoid Involute: $x = 2r \cos t + r \cos 2t$, $y = 2r \sin t - r \sin 2t$

where, r = radius of rolling circle involved in formation of deltoid.

Parabola Involute: $x^3 = ay^2$.

1.1.7 Envelope

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

CASE 1: Envelope of one parameter family of curves.

Let us consider $y = f(x, \alpha)$ to be the given family of curves with 'α' as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter.

Step 2: By substituting the value of parameter α in the given family of curves, we get the required envelope.

SPECIAL CASE: If the given equation of curve is quadratic in terms of parameter, *i.e.* $A\alpha^2 + B\alpha + c = 0$, then the envelope is given by **discriminant = 0** *i.e.* $B^2 - 4AC = 0$.

CASE 2: Envelope of two parameter family of curves.

Let us consider $y = f(x, \alpha, \beta)$ to the given family of curves, and a relation connecting the two parameters α and β, $g(\alpha, \beta) = 0$.

Step 1: Consider α as independent variable and β depends on α. Differentiate $y = f(x, \alpha, \beta) = 0$ and $g(\alpha, \beta) = 0$ w.r.t. the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and $g(\alpha, \beta) = 0$, we get the required envelope.

SOME SOLVED EXAMPLES

Example 1.11. *Show that the centre of curvature and equation of circle of curvature at the point* $\left(\frac{a}{4}, \frac{a}{4}\right)$ *on the curve* $\sqrt{x} + \sqrt{y} = \sqrt{a}$ *is* $\left(\frac{3}{4}a, \frac{3}{4}a\right)$ *and* $\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \frac{a^2}{2}$.

Solution. Here, the equation of the curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \qquad \text{...}(i)$$

Differentiating (i) w.r.t. 'x', we get

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\, y_1 = 0 \qquad \text{...}(ii)$$

Differentiating (ii) w.r.t. 'x', we get

$$-\frac{1}{4}x^{-3/2} - \frac{1}{4}y^{-3/2} \,.\, y_1 \,.\, y_1 + \frac{1}{2}y^{-1/2} \,.\, y_2 = 0 \qquad \text{...}(iii)$$

From (ii), at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$, we have

$$\frac{1}{2}\cdot\frac{2}{\sqrt{a}} + \frac{1}{2}\cdot\frac{2}{\sqrt{a}}\, y_1 = 0$$

$$\Rightarrow \qquad y_1 = -1$$

From equation (iii), at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$

$$-\frac{1}{4}\cdot\frac{4}{a}\cdot\frac{2}{\sqrt{a}} - \frac{1}{4}\cdot\frac{4}{a}\cdot\frac{2}{\sqrt{a}}(-1)^2 + \frac{1}{2}\cdot\frac{2}{\sqrt{a}}\, y_2 = 0$$

$$\Rightarrow \qquad -\frac{4}{a\sqrt{a}} + \frac{1}{\sqrt{a}}\, y_2 = 0$$

$$\Rightarrow \qquad y_2 = \frac{4}{a}$$

Now, ρ (at the given point) $= \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+1)^{3/2}}{4/a}$

$$= 2\sqrt{2}\,\frac{a}{4} = \frac{a}{\sqrt{2}}$$

Let $(\bar{x}, \bar{y})$ be the centre at curvature at $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= \frac{a}{4} - \frac{-(1+1)}{4/a}$$

$$= \frac{a}{4} + \frac{a}{2} = \frac{3}{4}a$$

Similarly, $\bar{y} = y + \dfrac{1+y_1^2}{y_2}$

$$= \frac{a}{4} + \frac{1+1}{4/a}$$

$$= \frac{a}{4} + \frac{a}{2} = \frac{3}{4}a$$

∴ Equation of the circle at curvature is,

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

i.e. $\left(x - \dfrac{3}{4}a\right)^2 + \left(y - \dfrac{3}{4}a\right)^2 = \dfrac{1}{2}a^2$. **(Proved)**

Example 1.12. *Show that the equation of the evolute of the tractrix $x = c\cos t + c\log\tan\dfrac{t}{2}$, $y = c\sin t$ is the catenary $y = c\cosh\dfrac{x}{c}$.*

Solution. Given equation of curve is

$$x = c\cos t + c\log\tan\frac{t}{2},\ y = c\sin t$$

Differentiating w.r.t. 't', we have

$$\frac{dx}{dt} = -c\sin t + \frac{c}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}$$

$$= -c\sin t + \frac{c\cos\frac{t}{2}}{\sin\frac{t}{2}}\cdot\frac{1}{2\cos^2\frac{t}{2}}$$

$$= -c\sin t + \frac{c}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -c\sin t + \frac{c}{\sin t}$$

$$= \frac{c(1-\sin^2 t)}{\sin t} = \frac{c\cos^2 t}{\sin t}$$

and $$\frac{dy}{dt} = c\cos t$$

Thus, $$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= c\cos t\,.\,\frac{\sin t}{c\cos^2 t} = \tan t$$

and $$y_2 = \frac{d^2y}{dx^2} = \sec^2 t\,.\,\frac{dt}{dx}$$

$$= \frac{1}{\cos^2 t}\cdot\frac{\sin t}{c\cos^2 t}$$

$$= \frac{\sin t}{c\cos^4 t}$$

Let $(\bar{x}, \bar{y})$ is the centre of curvature at any point on the curve, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= c\cos t + c\log\tan\frac{t}{2} - c\,\frac{\cos^4 t}{\sin t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos^2 t}$$

$$= c\cos t + c\log\tan\frac{t}{2} - c\cos t$$

or $$\bar{x} = c\log\left(\tan\frac{t}{2}\right) \qquad ...(i)$$

and $$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= c\sin t + \frac{1+\tan^2 t}{\dfrac{\sin t}{c\cos^4 t}}$$

$$= c\sin t + \frac{c\cos^4 t}{\sin t}\,.\,\sec^2 t$$

$$= c\sin t + \frac{c\cos^2 t}{\sin t}$$

$$= \frac{c(\sin^2 t + \cos^2 t)}{\sin t}$$

$$\bar{y} = \frac{c}{\sin t} \quad \{\because \quad \sin^2 t + \cos^2 t = 1\} \qquad ...(ii)$$

Evolute of the given curve is the locus of $(\bar{x}, \bar{y})$. Let us eliminate 't' between (i) and (ii).

From (i), $$\log\tan\frac{t}{2} = \frac{\bar{x}}{c}$$

$$\Rightarrow \qquad \tan\frac{t}{2} = e^{\bar{x}/c}$$

From (ii), $$\frac{\bar{y}}{c} = \frac{1}{\sin t} = \frac{1+\tan^2 t/2}{2\tan t/2}$$

$$\frac{\bar{y}}{c} = \frac{1}{2}\left(\frac{1}{\tan t/2} + \tan\frac{t}{2}\right)$$

$$= \frac{1}{2}\left(e^{-\frac{\bar{x}}{c}} + e^{\frac{\bar{x}}{c}}\right)$$

$$\Rightarrow \qquad \bar{y} = c\cosh\frac{\bar{x}}{c}$$

Changing $\bar{x}$ to x, $\bar{y}$ to y, the locus at $(\bar{x}, \bar{y})$ is

$$y = c\cosh\frac{x}{c},$$

which is the equation of evolute.

Example 1.13. *Find the coordinates of centre of curvature at any point for the given curve $x^{2/3} + y^{2/3} = a^{2/3}$. Also find the values of radius of curvature (ρ) and equation of evolute.*

Solution. The parametric equation of given curve is

$$x = a\cos^3 t, \qquad y = a\sin^3 t \qquad \text{...(1)}$$

Differentiating w.r.t. 't', the above equation gives,

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore \qquad y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

$$\therefore \qquad y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(-\tan t)$$

$$= -\sec^2 t\,\frac{dt}{dx}$$

$$= \frac{-\sec^2 t}{-3a\cos^2 t\sin t} = \frac{1}{3a\sin t\cos^4 t}$$

$$\therefore \quad \text{Radius of curvature } (\rho) = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= (1+\tan^2 t)^{3/2} \times 3a\sin t\cos^4 t$$

$$= \frac{3a \sin t \cos^4 t}{(\cos^2 t)^{3/2}}$$
$$= 3a \sin t \cos t$$

Now, the centre of curvature $(\bar{x}, \bar{y})$ at the point 't', is given by

We have, $\bar{x} = x - \rho \sin \psi$

$$= x - \frac{y_1(1 + y_1^2)}{y_2}$$
$$= x + \tan t\,(1 + \tan^2 t) \times 3a \sin t \cos^4 t$$

$\Rightarrow$ $\bar{x} = x + 3a \sin^2 t \cos t$

$$= a \cos^3 t + 3a \sin^2 t \cos t \qquad ...(2)$$

$[\because \quad x = a \cos^3 t]$

Similarly, $$\bar{y} = y + \frac{(1 + y_1^2)}{y_2}$$

$$\bar{y} = y + \frac{(1 + \tan^2 t)}{\frac{1}{3a \sin t \cos^4 t}}$$
$$= y + (1 + \tan^2 t)\,(3a \sin t \cos^4 t)$$
$$= y + 3a \sin t \cos^2 t$$
$$= a \sin^3 t + 3a \sin t \cos^2 t \qquad ...(3)$$

Eliminating x, y and t from equation (1), (2) and (3), we get

$$\bar{x} + \bar{y} = a\,(\cos^3 t + \sin^3 t + 3 \sin^2 t \cos t + 3 \cos^2 t \sin t)$$
$$= a(\cos t + \sin t)^3$$

or $$(\bar{x} + \bar{y})^{2/3} = a^{2/3}\,(\cos t + \sin t)^2 \qquad ...(4)$$

Similarly, $$(\bar{x} - \bar{y})^{2/3} = a^{2/3}\,(\cos t - \sin t)^2 \qquad ...(5)$$

Adding (4) and (5), we have

$$(\bar{x} + \bar{y})^{2/3} + (\bar{x} - \bar{y})^{2/3} = a^{2/3}\,(\cos^2 t + \sin^2 t + 2 \cos t \sin t + \cos^2 t + \sin^2 t - 2 \cos t \sin t)$$
$$= a^{2/3}(2) = 2a^{2/3}$$

So, the locus of $(\bar{x}, \bar{y})$ *i.e.*, the equation of evolute is

$$(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$$

and the coordinates of the centre of curvature are $[(a \cos^3 t + 3 \sin^2 t \cos t), (a \sin^3 t + 3 \sin t \cos^2 t)]$.

Example 1.14. *For the given rectangular hyperbola $xy = a^2$ (i.e. $x = at$, $y = a/t$)*

i. *find the radius of curvature (ρ)*

ii. *find the coordinates of centre of curvature (i.e. $\bar{x}, \bar{y}$)*

iii. *shows that the evolute of given curve is $(x + y)^{2/3} - (x - y)^{2/3} = (4a)^{2/3}$.*

Solution. i. To find radius of curvature, we have

$$xy = a^2, \qquad \text{or} \quad y = \frac{a^2}{x}$$

Therefore, $\quad y_1 = \dfrac{dy}{dx} = \dfrac{-a^2}{x^2}$ and $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{2a^2}{x^3}$

Thus,
$$\rho = \frac{(1+y_1{}^2)^{3/2}}{y_2}$$
$$= \frac{\left(1+\dfrac{a^4}{x^4}\right)^{3/2}}{\dfrac{2a^2}{x^3}}$$

$$\Rightarrow \qquad \rho = \frac{x^3}{2a^2}\left(\frac{x^4+a^4}{x^4}\right)^{3/2}$$

ii. To find coordinates of centre of curvatures, we have

$$\bar{x} = x - \frac{y_1(1+y_1{}^2)}{y_2} \quad \text{and} \quad \bar{y} = y + \frac{1+y_1{}^2}{y_2}$$

$$\Rightarrow \qquad \bar{x} = x - \frac{\dfrac{-a^2}{x^2}\left(1+\dfrac{a^4}{x^4}\right)}{\dfrac{2a^2}{x^3}}$$
$$= x + \frac{x^4+a^4}{2x^3}$$
$$= \frac{2x^4+x^4+a^4}{2x^3}$$
$$= \frac{3x^4+a^4}{2x^3} \qquad \text{[Since } xy = a^2\text{]}$$

We have $\quad \dfrac{3x^4+x^2y^2}{2x^3} = \dfrac{3}{2}x + \dfrac{y^2}{2x}$

Similarly,
$$\bar{y} = y + \frac{1+y_1{}^2}{y_2}$$
$$= y + \frac{1+\dfrac{a^4}{x^4}}{\dfrac{2a^2}{x^3}}$$
$$= y + \frac{x^4+a^4}{2a^2x}$$
$$= y + \frac{x^4+x^2y^2}{2x^2y} \qquad \text{[Since } xy = a^2\text{]}$$

$$= y + \frac{x^2+y^2}{2y} = \frac{x^2+3y^2}{2y} = \frac{3}{2}y + \frac{x^2}{2y}$$

Thus, the coordinates of the centre of curvature are

$$(\bar{x}, \bar{y}) = \left(\frac{3x}{2} + \frac{y^2}{2x}, \frac{3y}{2} + \frac{x^2}{2y}\right)$$

iii. Further, to show the equation of the evolute of the given curve is $(x + y)^{2/3} - (x - y)^{2/3} = (4a)^{2/3}$, we have

$$(\bar{x} + \bar{y}) = \frac{1}{2xy}[x^3 + y^3 + 3x^2y + 3xy^2]$$

$$= \frac{1}{2xy}(x+y)^3 = \frac{1}{2a^2}(x+y)^3$$

and do, $$(\bar{x} + \bar{y})^{2/3} = \frac{1}{(2a^2)^{2/3}}(x+y)^2 \qquad \{\because \quad xy = a^2]$$

Similarly, $$(\bar{x} - \bar{y})^{2/3} = \frac{1}{(2a^2)^{2/3}}(x-y)^2$$

Therefore, we have

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = \frac{1}{(2a^2)^{2/3}}[(x+y)^2 - (x-y)^2]$$

$$= \frac{1}{(2a^2)^{2/3}}(4xy) = \frac{1}{(2a^2)^{2/3}}(4a^2) = (4a)^{2/3}$$

Thus, the locus of $(\bar{x}, \bar{y})$ is

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = (4a)^{2/3}.$$

Example 1.15. *Find the centre of curvature of the following curves:*

i. $y = x^3 - 6x^2 + 3x + 1$ *at* $(1, -1)$

ii. *the parabola* $y^2 = 4ax$ *at* (x, y). *Also find the equation of the evolute of the given curve.*

iii. *the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at* (x, y). *Also find its evolute.*

Solution. i. The given curve is

$$y = x^3 - 6x^2 + 3x + 1$$

Therefore, $$y_1 = \frac{dy}{dx} = 3x^2 - 12x + 3$$

so $$y_1 \text{ at } (1, -1) = -6$$

and $$y_2 = \frac{d^2y}{dx^2} = 6x - 12$$

$\therefore$ $$y_2 \text{ at } (1, -1) = -6$$

Thus, $$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 1 - \frac{(-6)[1+(-6)^2]}{-6} = -36$$

Similarly, $$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= -1 + \frac{1+(-6)^2}{-6} = -\frac{43}{6}$$

Hence the centre of curvature are $(\bar{x}, \bar{y}) = (-36, -43/6)$.

ii. The given equation of the parabola is $y^2 = 4ax$.

Therefore, $$y = 2\sqrt{ax} \quad \Rightarrow \quad y_1 = 2\sqrt{a}\,.\frac{1}{2\sqrt{x}} = \sqrt{\frac{a}{x}}$$

and $$y_2 = \frac{d^2y}{dx^2} = -\frac{1}{2}\sqrt{a}\,x^{-3/2}$$

Therefore, $$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{\sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right)}{-\frac{1}{2}\sqrt{a}x^{-3/2}} = x + 2(x+a)$$

$$= 3x + 2a \qquad \text{...(1)}$$

and $$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{\left(1+\frac{a}{x}\right)}{-\frac{1}{2}\sqrt{a}x^{-3/2}}$$

$$= 2\sqrt{a}\sqrt{x} - \frac{2(x+a)}{\sqrt{a}x^{-1/2}} \qquad \{\because \quad y = 2\sqrt{ax}\ \}$$

$$= 2\sqrt{a}\sqrt{x}\left(1 - \frac{x-a}{a}\right)$$

$$= -2\,\frac{x^{3/2}}{\sqrt{a}} \qquad \text{...(2)}$$

Hence the centre of curvature of the given curve is

$$(\bar{x}, \bar{y}) = \left(3x + 2a, -\frac{2x^{3/2}}{\sqrt{a}}\right)$$

From (1), we have $$x = \frac{\bar{x} - 2a}{3}$$

Putting this value in (2), we have

$$\bar{y} = -\frac{2\left(\frac{\bar{x}-2a}{3}\right)^{3/2}}{\sqrt{a}}$$

or $$a\bar{y}^2 = 4\left(\frac{\bar{x}-2a}{3}\right)^3$$

or $$27\,a\bar{y}^2 = 4(\bar{x}-2a)^3$$

Therefore, the locus of the centre of curvature $(\bar{x}, \bar{y})$ is $27ay^2 = 4(x-2a)^3$, which is the required evolute.

iii. The given equation of the ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

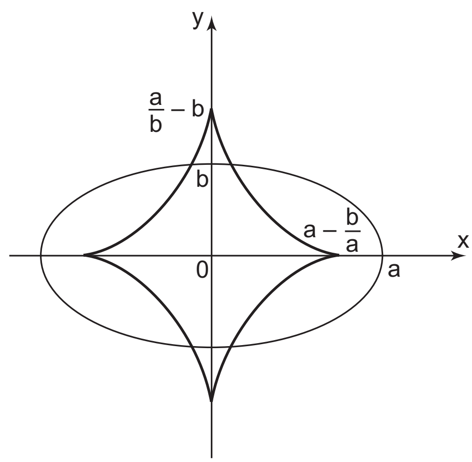


Fig. 1.16

Therefore, $y_1 = \dfrac{dy}{dx} = \dfrac{-b^2x}{a^2y}$ and $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{-b^4}{a^2y^3}$

Therefore, $(\bar{x}, \bar{y})$ centre of curvature are,

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{-\frac{b^2x}{a^2y}\left(1+\frac{b^4x^2}{a^4y^2}\right)}{-\frac{b^4}{a^2y^3}}$$

$$= x - \frac{x}{a^4b^2}(a^4y^2 + b^4x^2)$$

$$= x - \frac{x}{a^4b^2}\,[a^2b^2(a^2-x^2) + b^4x^2]$$

$$= \frac{a^2-b^2}{a^4}x^3 \qquad \text{...(1)}$$

Similarly, $$\bar{y} = y + \left(\frac{1+y_1^2}{y_2}\right)$$

$$= y + \frac{1+\frac{b^4x^2}{a^4y^2}}{-\frac{b^4}{a^2y^3}}$$

$$= y - \frac{y}{a^2b^4}\,(a^4y^2 + b^4x^2)$$

$$= y - \frac{y}{a^2b^4}\,[a^4y^2 + b^2a^2(b^2-y^2)]$$

$$= \frac{b^2 - a^2}{b^4} y^3 \qquad \ldots(2)$$

From eqn. (1), we have

$$x = \left(\frac{a^4 \bar{x}}{a^2 - b^2} \right)^{1/3}$$

and from eqn. (2), we have

$$y = \left(\frac{b^4 \bar{y}}{b^2 - a^2} \right)^{1/3}$$

Substituting these values in equation of ellipse, we have

$$\frac{1}{a^2}\left(\frac{a^4 \bar{x}}{a^2 - b^2} \right)^{2/3} + \frac{1}{b^2}\left(\frac{b^4 \bar{y}}{b^2 - a^2} \right)^{2/3} = 1$$

or $\qquad (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$

Therefore the locus of the centre of curvature $(\bar{x}, \bar{y})$ is

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$$

which is the required evolute for the given ellipse.

Example 1.16. *Find the envelope of the family of straight line* $y = mx + \sqrt{a^2m^2 + b^2}$ *, m is the parameter.*

Solution. Given equation of family of curves

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$\Rightarrow \qquad (y - mx) = \sqrt{a^2m^2 + b^2}$

$\Rightarrow \qquad (y - mx)^2 = (a^2m^2 + b^2)$

$\Rightarrow \qquad y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$

$\Rightarrow \qquad m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$

Differentiate partially w.r.t the parameter (*i.e.* m)

$\Rightarrow \qquad 2m(x^2 - a^2) - 2xy = 0$

$\Rightarrow \qquad m = \dfrac{xy}{(x^2 - a^2)}$

Substituting the value of m in the given family of curves

$\Rightarrow \qquad m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$

$\Rightarrow \qquad \left(\dfrac{xy}{(x^2 - a^2)} \right)^2 (x^2 - a^2) - 2\dfrac{xy}{(x^2 - a^2)} xy + (y^2 - b^2) = 0$

$\Rightarrow \qquad \dfrac{x^2y^2}{x^2 - a^2} - \dfrac{2x^2y^2}{x^2 - a^2} + (y^2 - b^2) = 0$

$\Rightarrow \qquad -\dfrac{x^2y^2}{x^2 - a^2} + (y^2 - b^2) = 0$

$\Rightarrow \qquad \dfrac{x^2y^2}{x^2 - a^2} = (y^2 - b^2)$

$\Rightarrow \quad x^2y^2 = (x^2 - a^2)(y^2 - b^2)$

$\Rightarrow \quad x^2y^2 = x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2$

$\Rightarrow \quad x^2b^2 + a^2y^2 = a^2b^2$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ The envelope of the given family of straight lines is an ellipse.

Example 1.17. *Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants.*

Solution. Given, $y = mx + am^p$ ...(1)

Differentiate the equation (1) w.r.t. parameter m, we get

$$0 = x + pam^{p-1}$$

$$\Rightarrow \quad m = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} \quad ...(2)$$

Using (2) and eliminates m from (1)

$$y = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} x + a\left(\frac{-x}{pa}\right)^{\frac{p}{p-1}}$$

$$\Rightarrow \quad y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1}\left(\frac{-x}{pa}\right)^{p}$$

i.e. $apy^{p-1} = (-x)^p + a^{p-2}(-x)^p$

which is the required equation of envelope of (1).

Problems Based on Envelope of Two Parameter Family of Curves:

Example 1.18. *Find the envelope of family of straight lines $ax + by = 1$, where a and b are parameters connected by the relation $ab = 1$.*

Solution. Given, $ax + by = 1$ ...(1)

and $ab = 1$ ...(2)

Differentiating (1) w.r.t 'a' (considering 'a' as independent variable and 'b' depends on a).

$$x + \frac{db}{da}y = 0$$

i.e. $$\frac{db}{da} = \frac{-x}{y} \quad ...(3)$$

Differentiating (2) w.r.t 'a'

$$b + a\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b}{a} \quad ...(4)$$

From (3) and (4), we have

$$\frac{x}{y} = \frac{b}{a}$$

i.e. $$\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$$

$\therefore$ $$a = \frac{1}{2x} \quad \text{and} \quad b = \frac{1}{2y} \qquad ...(5)$$

Using (5) in (2), we get the envelope as $4xy = 1$.

Example 1.19. *Find the envelope of family of curve* $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$*, where a and b are parameters connected by the relation* $\sqrt{a} + \sqrt{b} = 1$.

Solution. Given, $$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \qquad ...(1)$$

and $$\sqrt{a} + \sqrt{b} = 1 \qquad ...(2)$$

Differentiating (1) with respect to 'a'

$$\frac{\sqrt{x}}{-2a^{3/2}} + \frac{\sqrt{y}}{-2b^{3/2}}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{x}\,b^{3/2}}{\sqrt{y}\,a^{3/2}} \qquad ...(3)$$

Differentiating (2) with respect to 'a'

$$\frac{1}{2\sqrt{a}} + \frac{1}{2\sqrt{b}}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{b}}{\sqrt{a}} \qquad ...(4)$$

From (3) and (4), we have

$$\frac{\sqrt{x}}{\sqrt{y}}\frac{b}{a} = 1$$

i.e. $$\frac{\sqrt{\frac{x}{a}}}{\sqrt{a}} = \frac{\sqrt{\frac{y}{b}}}{\sqrt{b}} = \frac{\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}}{\sqrt{a} + \sqrt{b}} = \frac{1}{1}$$

$\therefore$ $$a = \sqrt{x} \text{ and } b = \sqrt{y} \qquad ...(5)$$

Using (5) in (2), we get the envelope as $x^{1/4} + y^{1/4} = 1$.

Example 1.20. *Find the envelope of family of straight line* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*, where a, b are two parameters which are connected by the relation* $a + b = c$.

Solution. Given equation of family of straight lines is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(1)$$

Also given, $$a + b = c$$

$\Rightarrow$ $$b = c - a \qquad ...(2)$$

Substituting (2) in (1), we get

$$\frac{x}{a}+\frac{y}{c-a} = 1$$

Differentiate w.r.t. a partially, we get

$$-\frac{x}{a^2}+\frac{y}{(c-a)^2} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} = \frac{y}{(c-a)^2}$$

$$\Rightarrow \quad \frac{(c-a)^2}{a^2} = \frac{y}{x}$$

$$\Rightarrow \quad \left(\frac{c-a}{a}\right)^2 = \frac{y}{x}$$

$$\Rightarrow \quad \frac{c-a}{a} = \sqrt{\frac{y}{x}}$$

$$\Rightarrow \quad \frac{c}{a}-1 = \sqrt{\frac{y}{x}}$$

$$\Rightarrow \quad \frac{c}{a} = 1+\sqrt{\frac{y}{x}}$$

$$\Rightarrow \quad \frac{c}{a} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$$

$$\Rightarrow \quad a = \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

Now, substitute the value of a in $b = c - a$

$$\Rightarrow \quad b = c - \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{c\sqrt{x}+c\sqrt{y}-c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \quad b = \frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Now, substitute the values of a and b in the given family of curves $\frac{x}{a}+\frac{y}{b} = 1$, we get

$$\frac{x}{\left(\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}\right)}+\frac{y}{\left(\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)} = 1$$

$$\Rightarrow \quad \frac{x(\sqrt{x}+\sqrt{y})}{c\sqrt{x}}+\frac{y(\sqrt{x}+\sqrt{y})}{c\sqrt{y}} = 1$$

$$\Rightarrow \qquad \frac{x(\sqrt{x}+\sqrt{y})}{\sqrt{x}}+\frac{y(\sqrt{x}+\sqrt{y})}{\sqrt{y}} = c$$

$$\Rightarrow \qquad (\sqrt{x}+\sqrt{y})\left(\frac{x}{\sqrt{x}}+\frac{y}{\sqrt{y}}\right) = 0$$

$$\Rightarrow \qquad (\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y}) = c$$

$$\Rightarrow \qquad (\sqrt{x}+\sqrt{y})^2 = c$$

$$\Rightarrow \qquad (\sqrt{x}+\sqrt{y}) = \sqrt{c} \text{ is the required envelope.}$$

Problems Based on Evolute as Envelope of its Normals:

Example 1.21. *By considering the evolute of a curve as the envelope of its normal, find the evolute of*

$$x = cos\ \theta + \theta\ sin\ \theta,\ y = sin\ \theta - \theta\ cos\ \theta.$$

Solution. Given $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta$$

Equation of normal line to the hyperbola is

$$(y-(\sin\theta-\theta\cos\theta)) = \frac{-1}{\tan\theta}(x-(\cos\theta+\theta\sin\theta))$$

$\Rightarrow \quad y\sin\theta - \sin^2\theta + \theta\sin\theta\cos\theta = -x\cos\theta + \cos^2\theta + \theta\sin\theta\cos\theta$

i.e. $\qquad y\sin\theta + x\cos\theta = 1 \qquad$ ...(1)

Differentiating (1) with respect to the parameter θ, we have

$$y\cos\theta - x\sin\theta = 0 \qquad ...(2)$$

Multiplying (1) by $\cos\theta$ and (2) by $\sin\theta$ and then subtracting, we have

$$x = \cos\theta \qquad ...(3)$$

Similarly, we get, $\qquad y = \sin\theta \qquad$ ...(4)

Eliminating θ between (3) and (4) we get the required evolute as $x^2 + y^2 = 1$.

Example 1.22. *Determine the evolute of hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ *= 1 by considering it as an envelope of its normal.*

Solution. Try yourself.

EXERCISE 1.2

1. Find the radius of curvatures and the centre of curvatures for the curve $y = \tan x$ at the point where $x = \pi/4$.
2. Find the centre of the curvatures for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, and also obtain its evolute.

3. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.
4. Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is $x^2 + y^2 = a^2$.
5. Determine the envelope of $x \sin\theta - y\cos\theta = a\theta$, where θ being the parameter.
6. Find the envelope of $x\sec^2\theta + y\,\mathrm{cosec}^2\theta = a$, where θ is the parameter.
7. Find the envelope of family of straight lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where a, b are parameters, connected by the relation $a^2b^3 = c^5$.
8. Find the envelope of the family of circles whose centres lie on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and which passes through its centre.
9. Determine the equation of the envelope of family of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where the parameters '$a$' and '$b$' are connected by the relation $\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = 1$, l and m are non-zero constants.

Answers

1. $\dfrac{5\sqrt{5}}{4}, \left(\dfrac{\pi - 10}{4}, \dfrac{9}{4}\right)$

2. **Hint:** $x = a\sec\theta$, $y = b\tan\theta$ centre of curvature $\left(\dfrac{a^2 + b^2}{a\cos^3\theta}, \dfrac{-\sin^3\theta\,(a^2 + b^2)}{\cos^3\theta}\right)$

   evolute $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$.

5. $x = a(\sin\theta + \cos\theta)$, $y = a(\sin\theta - \theta\cos\theta)$
6. $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0$
7. $x^2y^3 = \dfrac{72}{3125}c^5$
8. $(x^2 + y^2)^2 = 4(a^2x^2 + b^2y^2)$
9. $\dfrac{x}{l} + \dfrac{y}{m} = 1$

INTERESTING FACT

Have you ever observed the machines or the toys that contains a winding key, such as those instrument playing monkeys? The inside spiral spring undergoes a motion in a "***circular involute***".

VIDEO REFERENCES

Curvature and Evolutes

USES OF ICT

http://kmr.csc.kth.se/wp/research/math-rehab/learning-object-repository/geometry-2/metric-geometry/euclidean-geometry/geometry/plane-curves/evolutes/

APPLICATIONS TO REAL LIFE

- They are used in mechanical industries, especially the teeth industries, where teeth of revolving machines and gears are made, to minimize the vibrations as much as possible.
- Scroll and Gas compressors are two such machines used to pump, compress or pressurize fluids. There shape is an application of this concept, which makes sure that they are efficient and less noisy.
- The road safety needs to be kept in mind while designing road curvatures, and similarly the size of grinding wheel also needs to be considered. The concept of curvature comes into play at that time.

1.2 EVALUATION OF DEFINITE AND IMPROPER INTEGRAL

1.2.1 Definite Integral

A definite integral is denoted by $\int_a^b f(x)\,dx$ where 'a' is called the lower limit of the integral and 'b' is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has anti-derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points *i.e.*, $F(b) - F(a)$. The definite integral has a unique value.

1.2.1.1 Definite Integral as the Limit of a Sum

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non-negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the x-axis.

When we evaluate this area, it is equal to

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \frac{1}{n} [f(a) + f(a+h) + \ldots + f(a+(n-1)\,h)]$$

where $$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression is known as the definition of definite integral as the limit of sum.

SOME SOLVED EXAMPLES

Example 1.23. *Find* $\int_0^2 (x^2 + 1)\,dx$ *as the limit of a sum.*

Solution. By definition

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \frac{1}{n} [f(a) + f(a+h) + \ldots + f(a+(n-1)h)],$$

where $$h = \frac{b-a}{n}$$

Therefore,
$$\begin{aligned}
\int_0^2 (x^2+1)\,dx &= 2\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+\dots+f\left(\frac{2(n-1)}{n}\right)\right]\\
&= 2\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{2^2}{n^2}+1\right)+\left(\frac{4^2}{n}+1\right)+\dots+\left(\frac{(2n-2)^2}{n^2}+1\right)\right]\\
&= 2\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{(1+1+\dots+1)}_{n\text{ times}}+\frac{1}{n^2}(2^2+4^2+\dots+(2n-2)^2\right]\\
&= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2^2}{n^2}(1^2+2^2+\dots+(n-1)^2\right]\\
&= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{4}{n^2}\,\frac{n(n-1)(2n-1)}{6}\right]\\
&= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2}{3}\,\frac{(n-1)(2n-1)}{n}\right]\\
&= 2\lim_{n\to\infty}\left[1+\frac{2}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]\\
&= 2\left[1+\frac{4}{3}\right]=\frac{14}{3}.
\end{aligned}$$

Example 1.24. *Evaluate* $\int_0^2 e^x dx$ *as the limit of a sum.*

Solution. By definition
$$\int_0^2 e^x dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0+e^{2/n}+e^{4/n}+\dots+e^{(2n-2)/n}\right]$$
Using the sum of n terms of a G.P., where $a = 1$, $r = e^{2/n}$,

We have,
$$\begin{aligned}
\int_0^2 e^x dx &= 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]\\
&= 2\lim_{n\to\infty}.\frac{1}{n}\left[\frac{e^2-1}{e^{2/n}-1}\right]\\
&= \frac{\dfrac{2(e^2-1)}{1}}{\lim\limits_{n\to\infty}\left(\dfrac{e^{2/n}-1}{2/n}\right).2} = e^2-1 \qquad \left[\text{using } \lim_{h\to 0}\frac{e^n-1}{h}=1\right]
\end{aligned}$$

FUNDAMENTAL THEOREM OF CALCULUS

1.2.2. First Fundamental Theorem of Integral Calculus

If $f(x)$ is defined in the interval $[a, b]$, then the definite integral of $f(x)$ is defined as

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where $\frac{d}{dx}F(x) = f(x)$

The definite integral defined above denotes the area bounded by the curve $y = f(x)$, the x-axis and two ordinates at $x = a$ and $x = b$.

SOME SOLVED EXAMPLES

Example 1.25. *Evaluate* $\int_0^2 x^2\,dx$.

Solution. Given, $\int_0^2 x^2\,dx = \left|\frac{x^3}{3}\right|_0^2 = \left[\frac{(2)^3}{3} - 0\right]$

$$= \frac{8}{3}.$$

Example 1.26. *Evaluate* $\int_0^{2\pi} \cos x\,dx$.

Solution. $\int_0^{2\pi} \cos x\,dx = |-\sin x|_0^{2\pi}$

$$= [-\sin 2\pi + \sin 0] = [-0 + 0] = 0$$

Example 1.27. *Evaluate* $\int_2^3 (x+1)\,dx$.

Solution. Given, $\int_2^3 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_2^3$

$$= \left[\left(\frac{(3)^2}{2} + 3\right) - \left(\frac{(2)^2}{2} + 2\right)\right]$$

$$= \left[\left(\frac{9}{2} + 3\right) - \left(\frac{4}{2} + 2\right)\right]$$

$$= \left[\left(\frac{9+6}{2}\right) - \left(\frac{4+4}{2}\right)\right]$$

$$= \left[\frac{15}{2} - \frac{8}{2}\right]$$

$$= \frac{7}{2}.$$

1.2.3 Second Fundamental Theorem of Integral Calculus

Second fundamental theorem of calculus states that if $f(x)$ is continuous in the interval $[a, b]$ and F is the indefinite integral of $f(x)$ on $[a, b]$, then

$$F'(x) = f(x)$$

Mathematically, if $F(x) = \int_a^x f(t)\, dt$

then, $F'(x) = f(x)$

> **Remark:** As anti-derivatives and derivatives are opposites to each other, if you derive the anti derivative of the function, you will get original function.

Example 1.28. *Solve the given with the help of 2nd Fundamental theorem of Integral Calculus*

$$F(x) = \int_0^{x^3} (t^2 + t)\, dt$$

Solution. Given $F(x) = \int_0^{x^3} (t^2 + t)\, dt$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^{x^3} = F(x^3) - F(0)$$

$$F(x) = \left(\frac{(x^3)^3}{3} - \frac{(x^3)^2}{2}\right) - \left(\frac{(0)^3}{3} - \frac{(0)^2}{2}\right)$$

$$F(x) = \frac{x^9}{3} - \frac{x^6}{2}$$

$$F'(x) = 3x^8 - 3x^5$$

$$F'(x) = 3x^2\,((x^3)^2 + (x^3))$$

$3x^2$ is the derivative of the upper limit x^3 and $((x^3)^2 + (x^3))$ is the same as $(t^2 + t)$.

By the end of this equation, we can see that the derivative of $F(x)$, which is the integral of $f(x)$, is equivalent to the original function $f(x)$. The functions of $F'(x)$ and $f(x)$ are extremely similar.

Example 1.29. *Solve the given* $F(x) = \int_0^{x^2} (t + 7)^{1/2}\, dt$ *with the help of 2nd Fundamental theorem of Integral Calculus.*

Solution. Given $F(x) = \int_0^{x^2} (t + 7)^{1/2}\, dt$

$$= \left[\frac{2(t+7)^{3/2}}{3}\right]_0^{x^2}$$

$$= F(x^2) - F(0)$$

$$F(x) = \left(\frac{2(x^2+7)^{3/2}}{3}\right) - \left(\frac{2(0+7)}{3}\right)^{3/2}$$

$$F(x) = \frac{2(x^2+7)^{3/2}}{3} - \frac{2(7)^{3/2}}{3}$$

$$F'(x) = 2x\,(x^2 + 7)^{1/2}$$

$2x$ is the derivative of the upper limit x^2 and $(x^2 + 7)^{1/2}$ is same as $(t + 7)^{1/2}$.

Example 1.30. *Solve the given* $F(x) = \int_{-3}^{\sqrt{x}} (3t^2 - 30)\,dt$ *with the help of 2nd Fundamental theorem of Integral Calculus.*

Solution. Given

$$F(x) = \int_{-3}^{\sqrt{x}} (3t^2 - 30)\,dt$$

$$= \left[t^3 - 30t\right]_{-3}^{\sqrt{x}} = F(\sqrt{x}) - F(-3)$$

$$F(x) = [(\sqrt{x})^3 - 30(\sqrt{x})] - [(-3)^3 - 30(-3)]$$

$$F'(x) = \frac{3}{2}x^{1/2} - \frac{15}{x^{1/2}}$$

$$F'(x) = \frac{1}{2\sqrt{x}}\,(3x - 30)$$

$\dfrac{1}{2\sqrt{x}}$ is the derivative of the upper limit $\sqrt{x}$ and $(3x - 30)$ is same as $(3t^2 - 30)$.

1.2.4 Properties of Definite Integrals

Here we define some properties of definite integrals which are very useful in evaluating them.

1. If $f_1(x)$ and $f_2(x)$ are continuous and bounded functions over the interval $[a, b]$ and k_1 and k_2 are two constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x)]\,dx = k_1\int_a^b f_1(x)\,dx + k_2\int_a^b f_2(x)\,dx$$

This is called linearity property.

2. $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Both sides are equal to $F(b) - F(a)$, it shows that variable in integration is dummy.

3. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

For $\quad F(b) - F(a) = -\{F(a) - F(b)\}$

4. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Here, 'c' is defined as $a < c < b$.

R.H.S. is equal to $F(c) - F(a) + F(b) - F(c)$ which is equal to $F(b) - F(a)$.

5. $$\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$$

This is known as invariance property and can be prove as on putting $a - x = t$, we have $-dx = dt$.

$$\int_0^a f(a - x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt \qquad \text{[By Property (3)]}$$

$$= \int_0^a f(x)\,dx \qquad \text{[By Property (2)]}$$

6. $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \qquad \text{if } f(x) \text{ is even } i.e.\ f(-x) = f(x)$$
$$= 0 \qquad \text{if } f(x) \text{ is odd } i.e.\ f(-x) = -f(x)$$

Proof. $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad ...(1)$$

Putting $x = -t$ in the first integral on the right, we have

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt \qquad \text{[By Prop. (3)]}$$
$$= \int_{0}^{a} f(-x)\,dx \qquad \text{[By Prop. (2)]}$$

Substituting this in (1), we have

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x) + f(-x)\}\,dx$$
$$= \begin{cases} 2\int_{0}^{a} \{f(x)\,dx, \text{if } f(-x) = f(x) \\ \quad 0, \quad \text{if } f(-x) = -f(x) \end{cases}$$

Function like x^4, cos x etc. for which $f(-x) = f(x)$ are even functions.

Functions like x^3, sin x etc. for which $f(-x) = -f(x)$ are called odd functions.

7. $$\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \qquad \text{if } f(2a - x) = f(x)$$
$$= 0, \qquad \text{if } f(2a - x) = -f(x)$$

The property can be proved in a manner similar to property (6).

SOME SOLVED EXAMPLES

Example 1.31. *Evaluate* $\int_{0}^{\pi/2} \log \sin x\,dx$.

Solution. Given $$I = \int_{0}^{\pi/2} \log \sin x\,dx$$

Then, $$I = \int_{0}^{\pi/2} \log \cos x\,dx \qquad \text{[By Property (5)]}$$

Adding the two value of I, we get

$$2I = \int_{0}^{\pi/2} (\log \sin x + \log \cos x)\,dx$$
$$= \int_{0}^{\pi/2} \{\log (2 \sin x \cos x) - \log 2\}\,dx$$
$$= \int_{0}^{\pi/2} \log \sin 2x\,dx - \frac{1}{2}\pi \log 2$$
$$= \frac{1}{2}\int_{0}^{\pi} \log \sin u\,du - \frac{1}{2}\pi \log 2, \text{ where } x = \frac{u}{2}$$
$$= \int_{0}^{\pi/2} \log \sin u\,du - \frac{1}{2}\pi \log 2, \qquad \text{[By Property (7)]}$$

$$= I - \frac{1}{2}\pi \log 2$$

Therefore, $$I = -\frac{1}{2}\pi \log 2,$$

$$\therefore \quad \int_0^{\pi/2} \log \sin x \, dx = -\frac{1}{2}\pi \log 2.$$

Example 1.32. *Evaluate* $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$.

Solution. Let $$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx \qquad \text{[By Property (5)]}$$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x \, dx}{\sec x + \tan x} - I$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sec x \tan x - \tan^2 x}{1} dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1) \, dx$$

$$= \pi[\sec x - \tan x + x]_0^{\pi} = \pi(\pi - 2)$$

or $$I = \frac{\pi}{2}(\pi - 2)$$

Therefore,

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \frac{\pi}{2}(\pi - 2).$$

Example 1.33. *Evaluate* $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

Solution. Let $$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

Then $$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx \quad [\because \ \sin(\pi - x) = \sin x, \cos(\pi - x) = -\cos x]$$

[By Property (5)]

Adding the two values at I, we get

$$2I = \int_0^{\pi} \frac{(\pi - x + x)\sin x}{1+\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi\Big[\tan^{-1}\cos x\Big]_0^{\pi}$$

$$= -\pi\left(-\frac{1}{4}\pi - \frac{1}{4}\pi\right)$$

$$2I = \frac{2}{4}\pi^2 \quad \Rightarrow \quad I = \frac{\pi^2}{4}$$

Therefore, $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$.

Example 1.34. *Show that* $\displaystyle\int_0^{\pi/2} \cos^3 2x\,.\,\sin^4 4x\,dx = 0.$

Solution. Let $\displaystyle I = \int_0^{\pi/2} \cos^3 2x\,.\,\sin^4 4x\,dx$

Putting $2x = t$, we get

$$I = \frac{1}{2}\int_0^{\pi} \cos^3 t\,.\,\sin^4 2t\,dt$$

$$= \frac{1}{2}\int_0^{\pi} 2^4\,.\cos^3 t\,.\,\sin^4 t\,.\cos^4 t\,dt$$

$$= 8\int_0^{\pi} \sin^4 t\,.\cos^7 t\,dt = 0 \qquad \text{[By Property (7)]}$$

Example 1.35. *Evaluate* $\displaystyle\int_0^{1} \cot^{-1}(1 - x + x^2)\,dx.$

Solution. The given integral can be written as

$$I = \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1+x(x-1)}\right)dx$$

$$= \int_0^1 \tan^{-1}\left\{\frac{x-(x-1)}{1+x(x-1)}\right\}dx$$

$$= \int_0^1 [\tan^{-1} x - \tan^{-1}(x-1)]\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx - \int_0^1 \tan^{-1}(x-1)\,dx$$

But $\displaystyle\int_0^1 \tan^{-1}(x-1)\,dx = \int_0^1 \tan^{-1}(1-x-1)\,dx$

$$= -\int_0^1 \tan^{-1} x\,dx \qquad \text{[By Property (5)]}$$

$$\therefore \qquad I = 2\int_0^1 \tan^{-1} x\,dx$$

Integrating it by parts, taking '1' as a second function, we have

$$I = 2\left[\tan^{-1} x\,.\,x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 2\,.\,1\,.\,\frac{\pi}{4} - \left[\log(1+x^2)\right]_0^1$$

$$= \frac{\pi}{2} - \log 2$$

Therefore,

$$\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \log 2\,.$$

EXERCISE 1.3

1. Evaluate the given definite integrals as limit of sums:

i. $\int_a^b x\,dx$

ii. $\int_0^5 (x+1)\,dx$

iii. $\int_2^3 x^2\,dx$

iv. $\int_1^4 (x^2-x)\,dx$

v. $\int_{-1}^1 e^x dx$

2. Evaluate the definite integrals:

i. $\int_{-1}^1 (x+1)\,dx$

ii. $\int_1^2 (4x^3-5x^2+6x+9)\,dx$

iii. $\int_0^{\pi/2} \cos 2x\,dx$

iv. $\int_0^{\pi/4} \tan x\,dx$

v. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

vi. $\int_2^3 \frac{dx}{x^2-1}$

vii. $\int_2^3 \frac{x\,dx}{x^2+1}$

viii. $\int_0^1 x\,e^{x^2}\,dx$

ix. $\int_0^{\pi/4} (2\sec^2 x + x^3 + 2)\,dx$

x. $\int_0^1 \left(x\,e^x + \sin\frac{\pi x}{4}\right) dx$

3. Evaluate $\int_0^{\pi} \log(1+\cos x)\,dx.$

4. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx.$

5. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx.$

6. Show that $\int_0^{\pi} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$

7. Evaluate $\int_0^{\pi} \sin^6 x \cos^7 x\,dx.$

Answers

1. i. $\dfrac{a}{\sqrt{2}}$ ii. $\dfrac{35}{2}$ iii. $\dfrac{19}{3}$ iv. $\dfrac{27}{2}$

v. $e-\dfrac{1}{e}$

2. i. 2 ii. $\dfrac{64}{3}$ iii. 0 iv. $\dfrac{1}{2}\log 2$

v. $\dfrac{\pi}{2}$ vi. $\dfrac{1}{2}\log\dfrac{3}{2}$ vii. $\dfrac{1}{2}\log 2$ viii. $\dfrac{1}{2}(e-1)$

ix. $\dfrac{\pi^4}{1024}+\dfrac{\pi}{2}+2$ x. $1+\dfrac{4}{\pi}-\dfrac{2\sqrt{2}}{\pi}$

3. $-\pi\log 2$ **4.** $\dfrac{\pi}{8}\log 2$ **5.** $\dfrac{1}{\sqrt{2}}\log(1+\sqrt{2})$ **7.** 0

1.2.5 Improper Integral

The integral $\int_a^b f(x)\,dx$ is an improper integral if

i. either the interval of integration $[a, b]$ is not finite *i.e.* either 'a' or 'b' or both 'a' and 'b' are infinite.
ii. or the integrand $f(x)$ is not bounded on $[a, b]$.
iii. neither the interval $[a, b]$is finite nor $f(x)$ is bounded over it.

1.2.6 Types of Improper Integral

Improper integral is of three types, which are explained as follows:

a. **Improper Integral of First Kind:**

The definite integral $\int_a^b f(x)\,dx$ is an improper integral of first kind if either 'a' or 'b' or both are infinite but $f(x)$ is bounded.

For example: $\int_1^\infty \dfrac{dx}{\sqrt{x}}, \int_{-\infty}^0 e^{2x}\,dx$ are improper integrals of first kind.

In this case, we define

i. $\int_a^\infty f(x)\,dx = \underset{t\to\infty}{\text{Lt}} \int_a^t f(x)\,dx, (t > a)$

The improper integral $\int_a^\infty f(x)\,dx$ will be convergent if the limit on the right hand side exists finitely and will be divergent if the limit is $+\infty$ or $-\infty$.

ii. $\int_{-\infty}^b f(x)\,dx = \underset{t\to-\infty}{\text{Lt}} \int_t^b f(x)\,dx \quad (t < b)$

The improper integral $\int_{-\infty}^{b} f(x)\,dx$ will be convergent if the limit on the right hand side exists finitely and will be divergent if the limit is $+\infty$ or $-\infty$.

iii. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$ [For every c]

$$= \underset{t_1 \to -\infty}{\text{Lt}} \int_{t_1}^{c} f(x)\,dx + \underset{t_2 \to \infty}{\text{Lt}} \int_{c}^{t_2} f(x)\,dx \qquad [t_1 < c < t_2]$$

The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and will be divergent if the limit is $+\infty$ or $-\infty$.

SOME SOLVED EXAMPLES

Example 1.36. *Examine the convergence of the improper integral* $\int_{1}^{\infty} \frac{dx}{\sqrt{x}}$.

Solution. Here
$$\int_{1}^{\infty} \frac{dx}{\sqrt{x}} = \lim_{t \to \infty} \int_{1}^{t} \frac{dx}{\sqrt{x}}$$
$$= \lim_{t \to \infty} \int_{1}^{t} x^{-1/2}\,dx$$
$$= \lim_{t \to \infty} \left[2x^{1/2}\right]_1^t$$
$$= \lim_{t \to \infty} [2\sqrt{t} - 2] = \infty$$

Hence, the given improper integral is divergent.

Example 1.37. *Solve the improper integral* $\int_{-\infty}^{0} e^{-x}\,dx$.

Solution. Here
$$\int_{-\infty}^{0} e^{-x}\,dx = \lim_{t \to -\infty} \int_{t}^{0} e^{-x}\,dx$$
$$= \lim_{t \to -\infty} \left[\frac{e^{-x}}{-1}\right]_t^0$$
$$= \lim_{t \to -\infty} [-(1 - e^{-t})]$$
$$= -1 + e^{\infty} = \infty$$

Hence, the given improper integral is divergent.

Example 1.38. *Check the convergence of the following improper integral* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution. Here
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$
$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{1+x^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} x\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} 0 - \tan^{-1} t_1\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} t_2 - \tan^{-1} 0\right]$$

$$= -\left[\tan^{-1}(-\infty)\right] + \left[\tan^{-1}(\infty)\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Hence the given improper integral is convergent.

b. **Improper Integral of Second Kind:**

The definite integral $\int_a^b f(x)\,dx$ is said to be improper integral of second kind if both 'a' and 'b' are finite and $f(x)$ is not bounded (*i.e.* $f(x)$ has one or more points of infinite discontinuity).

For example: $\int_0^1 \frac{1}{x}\,dx, \int_1^4 \frac{dx}{(x-1)(x-4)}$ are improper integrals of second kind. In this case, we define

i. $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$, if '$a$' is the only point of infinite discontinuity of $f(x)$.

If the limit on the R.H.S. exists finitely, then it is convergent otherwise it is divergent.

ii. $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$, if '$b$' is the only point of infinite discontinuity of $f(x)$.

If the limit on the R.H.S. exists finitely, then it is convergent, otherwise, it is divergent.

iii. If $f(x)$ becomes infinite at some point 'c' only with $a < c < b$, then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon_1 \to 0^+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0^+} \int_{c+\varepsilon_2}^b f(x)\,dx$$

In general, if $c_1, c_2, c_3, \ldots\ldots, c_n$ are some finite number of points of infinite discontinuity of $f(x)$ on $[a, b]$, where $a < c_1 < c_2 < c_3, \ldots\ldots < c_{n-1} < c_n < b$, then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots\ldots + \int_{c_n}^b f(x)\,dx$$

If the limit on R.H.S. exists finitely, then the improper integral will be convergent otherwise it will be divergent.

SOME SOLVED EXAMPLES

Example 1.39. *Test the convergence of the integral* $\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution. The given integral is of second kind and '0' is the point of infinite discontinuity on [0, 1].

Therefore,
$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0^+} \int_{0+\varepsilon}^1 x^{-1/2}\,dx$$

$$= \lim_{\varepsilon \to 0^+} \left[2\sqrt{x}\right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0^+} 2(1-\sqrt{\varepsilon}) = 2 \text{ (finite)}$$

Hence given integral is convergent and converges to 2.

Example 1.40. *Test the convergence of the integral* $\int_0^1 \frac{dx}{x^2 - 3x + 2}$

Solution. The given integral is of second kind and '1' is the only point of discontinuity of $f(x)$. Therefore, by definition,

$$\begin{aligned}
\int_0^1 \frac{dx}{x^2 - 3x + 2} &= \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dx}{x^2 - 3x + 2} \\
&= \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)(2-x)} \\
&= \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \left[\frac{1}{1-x} - \frac{1}{2-x}\right] && \text{[By Partial fraction]} \\
&= \lim_{\varepsilon \to 0^+} \left[-\log(1-x) + \log(2-x)\right]_0^{1-\varepsilon} \\
&= \lim_{\varepsilon \to 0^+} [-\log \varepsilon + \log(1+\varepsilon) - \log 2] \\
&= \lim_{\varepsilon \to 0^+} \left[\log \frac{1+\varepsilon}{\varepsilon} - \log 2\right] \\
&= \lim_{\varepsilon \to 0^+} \left[\log\left(1 + \frac{1}{\varepsilon}\right) - \log 2\right] \\
&= \log(\infty) - \log(2) = \infty
\end{aligned}$$

Hence, the given integral is divergent.

Example 1.41. *Test the convergence of the integral* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution. The given integral is of second kind and '0' is the only point of infinite discontinuity in $[-1, 1]$.

Therefore, by definition,

$$\begin{aligned}
\int_{-1}^1 \frac{dx}{x^2} &= \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} \\
&= \lim_{\varepsilon_1 \to 0^+} \int_{-1}^{0-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \to 0^+} \int_{\varepsilon_2}^1 \frac{dx}{x^2} \\
&= \lim_{\varepsilon_1 \to 0^+} \left[\frac{x^{-2+1}}{-1}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0^+} \left[\frac{x^{-2+1}}{-1}\right]_{\varepsilon_2}^1 \\
&= \lim_{\varepsilon_1 \to 0^+} \left[-x^{-1}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0^+} \left[-x^{-1}\right]_{\varepsilon_2}^1 \\
&= \lim_{\varepsilon_1 \to 0^+} \left[\frac{1}{\varepsilon_1} - 1\right] + \lim_{\varepsilon_2 \to 0^+} \left[-1 + \frac{1}{\varepsilon_2}\right] \\
&= \infty
\end{aligned}$$

Hence, the given integral is divergent.

c. **Improper Integral of third kind (or Mixed kind):**

The definite integral $\int_a^b f(x)\,dx$ is said to be improper integral of third kind if either '*a*' or '*b*' or both are infinite and $f(x)$ is also unbounded.

For example: $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx, \int_{-\infty}^{\infty} \frac{1}{1-x}\,dx$ are improper integrals of third kind.

EXERCISE 1.4

1. Examine the convergence of the following improper integrals and if convergent, find their values:

i. $\int_2^\infty \frac{dx}{x\log x}$ ii. $\int_a^\infty \frac{x\,dx}{1+x^2}$ iii. $\int_{-\infty}^0 \frac{dx}{p^2+q^2x^2}$ iv. $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

v. $\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx$ vi. $\int_0^\infty \cos x\,dx$

2. Examine the convergence of the following improper integrals and if convergent, find their values also.

i. $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$ ii. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ iii. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$ iv. $\int_0^4 \frac{dx}{x(4-x)}$

v. $\int_0^\pi \frac{dx}{\sin x}$ vi. $\int_0^1 \log x\,dx$

Answers

1. i. divergent ii. divergent iii. convergent; $\frac{\pi}{2pq}$ iv. convergent; $\frac{\pi}{4}$

v. convergent; $\frac{\pi}{4}+\frac{1}{2}\log 2$ vi. divergent

2. i. convergent; $\frac{8}{3}$ ii. convergent; $\frac{\pi}{3}$ iii. convergent; 2 iv. divergent

v. divergent vi. convergent; –1

1.2.7 Comparison tests for convergence of $\int_a^b f(x)\,dx$ at '*a*'

1.2.7.1 Comparison Test-I

Statement: If f and g are two positive functions such that $f(x) \le g(x)$ for all x in $(a, b]$ and '*a*' is the only infinite discontinuity on $[a, b]$, then

i. $\int_a^b g\,dx$ is convergent $\Rightarrow \int_a^b f\,dx$ is convergent

ii. $\int_a^b f dx$ is divergent $\Rightarrow \int_a^b g dx$ is divergent

Proof. Since $0 < f(x) \leq g(x)\ \forall\ x \in (a, b]$

$$\therefore \qquad \int_{a+\epsilon}^b f dx \leq \int_{a+\epsilon}^b g dx \quad \text{for } 0 < \varepsilon < b - a \qquad ...(1)$$

i. Let $\int_a^b g dx$ be convergent at 'a', then there exists a positive number M such that

$$\int_{a+\epsilon}^b g dx < M \quad \text{for } 0 < \varepsilon < b - a \qquad ...(2)$$

From (1) and (2), we get

$$\int_{a+\epsilon}^b f dx < M \quad \text{for } 0 < \varepsilon < b - a$$

$\Rightarrow \quad \int_a^b f dx$ is convergent at 'a'.

ii. Let $\int_a^b f dx$ be divergent at 'a'. Then $\int_{a+\epsilon}^b f dx$ is unbounded above and hence from (1), $\int_{a+\epsilon}^b g dx$ is unbounded above.

Hence, $\int_a^b g dx$ is divergent at a.

1.2.7.2 Comparison Test-II

Statement: If 'f' and 'g' are two positive functions on $(a, b]$, 'a' being the only point of infinite discontinuity such that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = l \neq 0$$

then the two integrals $\int_a^b f dx$ and $\int_a^b g dx$ converges or diverges together at 'a'.

Proof: As f, g are positive in $(a, b]$, therefore $\frac{f(x)}{g(x)} > 0 \quad \forall\ x \in (a, b]$

Hence $$\lim_{x \to a^+} \frac{f(x)}{g(x)} = l \geq 0$$

But $l \neq 0$ (given). Therefore $l > 0$.

Now choose a positive number ε such that $l - \varepsilon > 0$,

since $$\lim_{x \to a^+} \frac{f(x)}{g(x)} = l$$

$\Rightarrow \quad$ There exists a neighbourhood (a, c) of a $(a < c < d)$ such that

$$\left| \frac{f(x)}{g(x)} - l \right| < \epsilon, \forall\ x \in (a, c]$$

$$\Rightarrow \qquad -\varepsilon < \frac{f(x)}{g(x)} - l < \varepsilon, \quad \forall\ x \in (a, c]$$

$$\Rightarrow \qquad (l - \varepsilon)\, g(x) < f(x) < (l + \varepsilon)\, g(x) \quad \forall\ x \in (a, c]$$

Case I: Let $\int_a^b f dx$ converges to a:

$\Rightarrow \quad \int_a^c fdx$ converges at a. [$\because \quad a < c < b$, and $\int_c^b fdx$ is a proper integral]

$\Rightarrow \quad (l - \varepsilon) \int_a^c gdx$ converges at a. [By comparison test I]

$\Rightarrow \quad \int_a^b gdx$ converges at a.

Case II: Let $\int_a^b f(x)$ diverges at a:

$\Rightarrow \quad \int_a^c f(x)(dx)$ diverges at a. [$\because \quad a < c < b$ and $\int_c^b fdx$ is a proper integral]

$\Rightarrow \quad (l + \varepsilon) \int_a^c gdx$ diverges at a [By comparison test I]

$\Rightarrow \quad \int_a^b gdx$ diverges at a.

Similarly, it can be shown that if $\int_a^b gdx$ converges at 'a', then $\int_a^b fdx$ converges at 'a' and if $\int_a^b gdx$ diverges at 'a', then $\int_a^b fdx$ diverges at 'a'

Hence the theorem.

1.2.8 Important Theorems

i. The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if $n < 1$.

ii. The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent if and only if $n < 1$.

Proof: i. The given integral $\int_a^b \frac{dx}{(x-a)^n}$ is a proper integral for $n \le 0$ and therefore convergent. If $n > 0$, then it is an improper integral and 'a' is only point of infinite discontinuity.

Case I: If $n \neq 1$

$$\begin{aligned}
\int_a^b \frac{dx}{(x-a)^n} &= \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n}, \; 0 < \varepsilon < b - a \\
&= \lim_{\varepsilon \to 0^+} \left[\frac{(x-a)^{-n+1}}{-n+1} \right]_{a+\varepsilon}^b \\
&= \lim_{\varepsilon \to 0^+} \left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}} \right] \frac{1}{(1-n)} \\
&= \begin{cases} \dfrac{1}{(1-n)(b-a)^{n-1}}, & \text{finite if } n < 1 \\ \infty & \text{if } n > 1 \end{cases}
\end{aligned}$$

$\therefore \quad \int_a^b \frac{dx}{(x-a)^n}$ converges for $0 < n < 1$ and diverges for $n > 1$

Case II. If $n = 1$

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^n} &= \int_a^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b \frac{dx}{x-a} \\ &= \lim_{\varepsilon \to 0^+} [\log|x-a|]_{a+\varepsilon}^b \\ &= \lim_{\varepsilon \to 0^+} [\log(b-a) - \log \varepsilon] \\ &= \lim_{\varepsilon \to 0^+} \left(\log \frac{b-a}{\varepsilon}\right) = \infty\end{aligned}$$

$\therefore \quad \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n = 1$

Hence, $\int_a^b \frac{dx}{(x-a)^n}$ is convergent for $n < 1$ and divergent for $n \geq 1$.

ii. The given integral $\int_a^b \frac{dx}{(b-x)^n}$ is a proper integral for $n \leq 0$ and therefore converges. If $n > 0$, then it is an improper integral and 'b' is the only point of infinite discontinuity.

Case I: If $n \neq 1$

$$\begin{aligned}\int_a^b \frac{dx}{(b-x)^n} &= \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} \frac{dx}{(b-x)^n} \\ &= \lim_{\varepsilon \to 0^+} \left[\frac{(b-x)^{-n+1}}{-(-n+1)}\right]_a^{b-\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{n-1}[(b-x)^{-n+1}]_a^{b-\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{n-1}[\varepsilon^{1-n} - (b-a)^{1-n}] \\ &= \begin{cases} \infty & \text{if } n > 1 \\ \dfrac{1}{(1-n)(b-a)^{n-1}} & \text{if } n < 1 \end{cases}\end{aligned}$$

$\therefore \quad \int_a^b \frac{dx}{(b-x)^n}$ converges for $0 < n < 1$ and diverges for $n > 1$.

Case II: If $n = 1$

$$\begin{aligned}\int_a^b \frac{dx}{(b-x)^n} &= \int_a^b \frac{dx}{b-x} \\ &= \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} \frac{dx}{b-x} \\ &= \lim_{\varepsilon \to 0^+} [-\log|b-x|]_a^{b-\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} [-\log \varepsilon + \log(b-a)] = \infty \qquad [\because \ \log 0 = -\infty]\end{aligned}$$

$\therefore \quad \int_a^b \frac{dx}{(b-x)^n}$ diverges if $n = 1$

Hence, $\int_a^b \frac{dx}{(b-x)^n}$ is convergent for $n < 1$ and divergent for $n \geq 1$

Remark: $\int_0^1 \frac{1}{x^n}\,dx$ is convergent if $n < 1$ and divergent if $n \geq 1$.

SOME SOLVED EXAMPLES

Example 1.42. *Examine the convergence of the integral* $\int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$

Solution. Let $$I = \int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$$

Here, $f(x) = \frac{1}{x^{1/2}(1+x^2)}$ and 0 is only point of infinite discontinuity of $f(x)$ and $f(x) > 0$ in $(0, 1]$

Take $$g(x) = \frac{1}{x^{1/2}}$$

Now, $$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{1}{1+x^2} = 1$$ [which is finite and non-zero]

$\therefore$ By comparison test,
the integrals

$\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converges or diverges together.

But the integral $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1/2}}$ is convergent at $x = 0$ $\left[\because \ n = \frac{1}{2} < 1\right]$

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$ is also convergent.

Example 1.43. *Discuss the convergence of the integral* $\int_0^{\pi/2} \frac{\sin x}{x^{3/2}}\,dx$.

Solution. Let $$I = \int_0^{\pi/2} \frac{\sin x}{x^{3/2}}\,dx$$

Here, $f(x) = \frac{\sin x}{x^{3/2}}$ and '0' is the only point of infinite discontinuity of $f(x)$ and $f(x) > 0$ in $\left[0, \frac{\pi}{2}\right]$

Take $$g(x) = \frac{1}{x^{1/2}}$$

Now, $$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \neq 0, \infty.$$

$\therefore$ By comparison test, the integrals $\int_0^{\pi/2} f(x)\,dx$ and $\int_0^{\pi/2} g(x)\,dx$ converge or diverge together.

But $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ $\left[\because \; n = \frac{1}{2} < 1\right]$ is convergent at $x = 0$

i.e., $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{\sin x}{x^{3/2}}\,dx$ is also convergent at $x = 0$

1.2.9 Comparison Test for Convergence at ∞

1.2.9.1 Comparison Test I

If f and g are two positive functions such that $f(x) \le g(x)$ for all $x \ge a$, then

i. $\int_a^\infty f dx$ converges if $\int_a^\infty g dx$ converges

ii. $\int_a^\infty g dx$ diverges if $\int_a^\infty f dx$ diverges

Proof: Here f and g are two positive functions such that $f(x) \le g(x)$ for all $x \in [a, t]$

$$\therefore \qquad \int_a^t f(x)\,dx \le \int_a^t g\,dx \qquad \text{...(1)}$$

i. Let $\int_a^\infty g dx$ be convergent, so that there exists a positive number M.

such that
$$\int_a^t g dx < M \;\forall\; t \ge a \qquad \text{...(2)}$$

From (1) and (2), we have

$$\int_a^t f dx < M \;\forall\; t \ge a$$

Hence $\int_a^\infty f dx$ is convergent.

ii. Let $\int_a^\infty f dx$ be divergent

$\Rightarrow$ $\int_a^t f dx$ is not bounded above and hence from (1), $\int_a^t g dx$ is also not bounded above, consequently, $\int_a^\infty g dx$ is divergent.

1.2.9.2 Comparison Test II

If f and g are two positive functions on $[a, \infty)$ such that

i. $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = l \qquad \text{(where } l \text{ is non-zero and finite),}$$

then the two integrals $\int_a^\infty f dx$ and $\int_a^\infty g dx$ converge or diverge together

Proof. i. As $\frac{f(x)}{g(x)} > 0$ for all $x \ge a$ and $l \ne 0$ $\quad$ [$\because$ $f(x)$ and $g(x)$ are positive functions]

$\therefore \quad l > 0$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$

Since $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = l$, therefore there exists a number k such that

$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon \ \forall \ x \geq k$$

$$\Rightarrow \qquad l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \ \forall \ x \geq k > a$$

or $$(l - \varepsilon)\, g(x) < f(x) < (l + \varepsilon)\, g(x) \ \forall \ x \geq k > a$$

By comparison test I, if $\int_a^\infty f dx$ converges, then $\int_a^\infty g(x)\, dx$ converges and if $\int_a^\infty g(x)\, dx$ converges, then $\int_a^\infty f(x)\, dx$ converges.

Similarly, divergence of one implies the divergence of other.

Hence, the two integrals $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ converge or diverge together.

1.2.10 Important Theorem

Statement: The improper integral $\int_a^\infty \dfrac{dx}{x^n}$, $(a > 0)$ converges if and only if $n > 1$ and diverges for $n \leq 1$.

Proof:

$$\begin{aligned}
\int_a^\infty \frac{dx}{x^n} &= \lim_{t \to \infty} \int_a^t \frac{dx}{x^n} \\
&= \lim_{t \to \infty} \left[\frac{x^{-n+1}}{-n+1} \right]_a^t, \text{ if } n \neq 1 \\
&= \lim_{t \to \infty} \left[\frac{t^{1-n}}{1-n} - \frac{a^{1-n}}{1-n} \right], \text{ if } n \neq 1 \\
&= \begin{cases} \dfrac{-a^{1-n}}{1-n}, & \text{(which is finite) if } n > 1 \\ \infty & \text{if } n < 1 \end{cases}
\end{aligned}$$

Also when $n = 1$, we have

$$\begin{aligned}
\int_a^\infty \frac{1}{x^n} dx &= \int_a^\infty \frac{1}{x} dx = \lim_{t \to \infty} \int_a^t \frac{1}{x} dx \\
&= \lim_{t \to \infty} [\log x]_a^t \\
&= \lim_{t \to \infty} [\log t - \log a] \\
&= \log \infty - \log a = \infty
\end{aligned}$$

Hence, $\int_a^\infty \dfrac{dx}{x^n}$ converges if and only if $n > 1$ and diverges for $n \leq 1$.

Remark: $\int_a^\infty \dfrac{dx}{x^n} dx$ is convergent if $(a > 0)$ is convergent if $n > 1$ and divergent if $n \leq 1$.

SOME SOLVED EXAMPLES

Example 1.44. *Test the convergence of the integral* $\int_1^\infty \frac{x^3}{(1+x)^5}\,dx$.

Solution. Let $$I = \int_1^\infty \frac{x^3}{(1+x)^5}\,dx$$

Here, $$f(x) = \frac{x^3}{(1+x)^5} = \frac{x^3}{x^5\left(1+\frac{1}{x}\right)^5}$$

$\Rightarrow$ $$f(x) = \frac{1}{x^2\left(1+\frac{1}{x}\right)^5}$$

Take $$g(x) = \frac{1}{x^2}$$

Now, $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{\left(1+\frac{1}{x}\right)^5} = 1 \neq 0, \infty$$

$\therefore$ By comparison test, the integrals $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ converge or diverge together.

But the integral $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$ $[\because\ n = 2 > 1$

is convergent.

$\therefore$ The integral $\int_1^\infty \frac{x^3}{(1+x)^5}\,dx$ is convergent.

Example 1.45. *Discuss the convergence of the improper integral* $\int_1^\infty x^n \,.\, e^{-x}\,dx$.

Solution. Let $$I = \int_1^\infty x^n \,.\, e^{-x}\,dx$$

Here, $$f(x) = x^n \,.\, e^{-x}$$

Take $$g(x) = \frac{1}{x^2}$$

Now, $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = 0\ \forall\ n$$

Now, $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^2}$ is convergent. $[\because\ n = 2 > 1]$

$\therefore$ By comparison test, $\int_1^\infty x^n \,.\, e^{-x} dx$ is also convergent.

1.2.11 Absolute Convergence

The improper integral $\int_a^b f dx$ is said to be absolutely convergent if $\int_a^b |f| dx$ is convergent.

Example 1.46. *Test the convergence at* $\int_0^1 \frac{\sin 1/x}{\sqrt{x}} dx$.

Solution. Let
$$I = \int_0^1 \frac{\sin 1/x}{\sqrt{x}} dx$$

Here,
$$f(x) = \frac{\sin 1/x}{\sqrt{x}},$$

does not keep the same sign in the neighbourhood of '0' and '0' is the point of infinite discontinuity of 'f' in [0, 1].

Now,
$$|f(x)| = \left|\frac{\sin 1/x}{\sqrt{x}}\right| = \frac{|\sin 1/x|}{\sqrt{x}} \le \frac{1}{\sqrt{x}} = g(x) \text{ (say)} \qquad \left[\because \left|\sin\frac{1}{x}\right| \le 1\right]$$

But
$$\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}} dx \text{ is convergent at } x = 0. \qquad \left[\because n = \frac{1}{2} < 1\right.$$

$\therefore \quad \int_0^1 |f|$ is convergent at 0.

Hence the given integral $\int_0^1 \frac{\sin 1/x}{\sqrt{x}} dx$ converges absolutely at '0'.

EXERCISE 1.5

1. Discuss the convergence of the following integrals:

i. $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ ii. $\int_0^1 \frac{x^n}{1+x} dx$ iii. $\int_0^1 \frac{dx}{x^3(1+x^2)^5}$

2. Test the convergence of $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

3. Examine the convergence of the following integrals:

i. $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$ ii. $\int_0^1 \frac{\log x}{\sqrt{x}} dx$ iii. $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ iv. $\int_0^{\pi/4} \frac{dx}{\sqrt{\sin x}}$

4. Test the convergence of the following integrals:

i. $\int_0^\infty \frac{\sin^2 x}{x^2} dx$ ii. $\int_1^\infty \frac{\log x}{x^2} dx$ iii. $\int_1^\infty \frac{dx}{\sqrt{x}(1+x^n)}$ iv. $\int_1^\infty \frac{dx}{\sqrt{x}(2+x)}$

v. $\int_0^\infty e^{-x^2} dx$

[**Hint:** $e^{x^2} > x^2 \ \forall\ x \in \mathbb{R}$]

Answers

1. i. convergent ii. convergent if $n > -1$ iii. divergent
2. convergent
3. i. convergent ii. convergent iii. convergent iv. convergent
4. i. convergent ii. convergent iii. convergent, if $n > 1/2$ iv. convergent v. convergent

INTERESTING FACTS

- This concept is used in pharmacological research to find out the plasma drug concentration, that is what is the maximum drug concentration and when it occurs.
- The '**R**'-value of any drug, which calculates the ratio of two different quantities is measured using this concept.
- Engineers use integral to find the ***Centre of Mass*** of any object.
- An interesting relationship in calculus is that the derivative and the integral are inverse processes. They are reverse of each other and they are linked using "***The Fundamental Theorem of Calculus***".

VIDEO REFERENCES

Beta & Gamma Function

Improper Integrals (Cont.) 1

Improper Integrals (Cont.) 2

APPLICATIONS TO REAL LIFE

- Application in statistics and probability.
- Significance in quantum physics and economics, which is created on the basis of probability distribution
- It is even used to find average changes, volumes, error estimations and surface areas.
- The same concept is used in finding Kinetic energy as well.

HISTORY

Special functions occur quite frequently in mathematical analysis. Among the special functions, gamma function seemed to be widely used. The gamma function $\Gamma(x)$ is applied in exact sciences almost as the well known factorial symbol $x!$. It was introduced by the famous mathematician L. Euler (1729). Beta function was first studied by Euler and Legendre and was given its name by Jacques Binet.

1.3 BETA, GAMMA FUNCTIONS AND THEIR PROPERTIES

1.3.1 Gamma Function

For $n > 0$, the improper integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ is defined as Gamma function and denoted by $\Gamma(n)$ (read as gamma n). It is also known as Eulerian Integral of second kind. Thus,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, n > 0 \qquad ...(1)$$

Note that Gamma function plays an important role in evaluation of definite integrals.

1.3.1.1 Properties of Gamma Function

a. To show that $\Gamma(n+1) = n\,\Gamma(n)$

We have, $\Gamma(n+1) = \int_0^\infty e^{-x} x^{(n+1)-1} dx$ [by (1)]

$$= [-e^{-x} x^n]_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= 0 + n\,\Gamma(n)$$

Thus, $\Gamma(n+1) = n\,\Gamma(n)$ ...(2)

b. To show that $\Gamma(n) = (n-1)!$, where n is positive integer.

We have $\Gamma(n) = (n-1)\,\Gamma(n-1)$ [by (2)]

$= (n-1)\,(n-2)\,\Gamma(n-2)$ [by (2)]

$= (n-1)\,(n-2)\,(n-3)\,...3.2\,\Gamma(2)$ [by repeated application of (2)]

$= (n-1)\,(n-2)\,(n-3)...3.2.1\,\Gamma(1)$

$= (n-1)!\,\Gamma(1) = (n-1)!\int_0^\infty e^{-x}\,dx$ [by (1)]

$= (n-1)!\,[-e^{-x}]_0^\infty = (n-1)!$

Thus $\Gamma(n) = (n-1)!$, n is positive integer

> **Remark 1:** The formula $\Gamma(n+1) = n\,\Gamma(n)$ is called as recurrence formula for Gamma Function.
> **Remark 2:** Note that $\Gamma(1) = 1$ can find out in (b).

c. To show that, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Put $n = \frac{1}{2}$ in (1), we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}\,x^{-1+\frac{1}{2}}\,dx$$

Putting $x = v^2 \Rightarrow dx = 2v dv$, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-v^2}\,dv \qquad ...(4)$$

Writing u in place of v in (4), we obtain

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du \qquad ...(5)$$

Multiplying (4) and (5), we get

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv$$

Now let $u = r\cos\theta$, $v = r\sin\theta$ then $u^2 + v^2 = r^2$ and $\theta = \tan^{-1}\left(\frac{v}{u}\right)$. Also $du\,dv = rdr\,d\theta$ and r varies from 0 to ∞; θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

Hence, $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad ...(6)$$

d. To show that $\Gamma(0) = \infty$

From (2), we have $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

As $n \to 0$, $\Gamma(0) = \lim\limits_{n\to 0}\dfrac{\Gamma(1)}{n} = \lim\limits_{n\to 0}\dfrac{1}{n} = \infty$

Thus, $$\Gamma(0) = \infty \qquad ...(7)$$

Further note that $\Gamma(-1), \Gamma(-2), \Gamma(-3)$ etc. are also undefined. Hence Gamma function is continuous for any $n > 0$ and is discontinuous at $n = 0, -1, -2, \ldots$.

Thus, $\Gamma(n)$ is defined for all n, except for zero and negative integers.

e. To show that $\Gamma(n + 1) = (m + 1)^{n+1}(-1)^n \int_0^1 x^m (\log x)^n\, dx$

where n is a positive integer and $m > -1$.

We have $\int_0^1 x^m (\log x)^n\, dx$

Put $x = e^{-y}$, then $dx = -e^{-y}\,dy = -xdy$

$$\therefore \quad \int_0^1 x^m (\log x)^n\, dx = \int_0^{\infty} e^{-my}(-y)^n e^{-y}\, dy = (-1)^n \int_0^{\infty} y^n e^{-(m+1)y}\, dy$$

Put $(m + 1)y = u$, so, $dy = \dfrac{du}{m+1}$

$$\therefore \quad \int_0^1 x^m (\log x)^n\, dx = (-1)^n \int_0^{\infty} \frac{u^n}{(m+1)^n} . e^{-u} . \frac{du}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^{\infty} e^{-u} . u^{(n+1)-1}\, du$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\,\Gamma(n+1) \qquad \text{[by (1)]}$$

Thus, $$\Gamma(n + 1) = (m + 1)^{n+1}(-1)^n \int_0^1 x^m (\log x)^n\, dx \qquad ...(8)$$

where n is a positive integer and $m > -1$.

Imp. Formula: $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}, m, n > -1.$

SOME SOLVED EXAMPLES

Example 1.47. *Evaluate* $\int_0^{\infty} x^{1/4} e^{-\sqrt{x}} dx$.

Solution. We have $\quad I = \int_0^{\infty} e^{-\sqrt{x}} x^{1/4} dx$

Now put $\quad \sqrt{x} = u \Rightarrow x = u^2 \Rightarrow dx = 2\,udu$

$$\therefore \quad I = \int_0^{\infty} e^{-u}(u^2)^{1/4}\, 2u\, du = 2\int_0^{x} e^{-u} u^{3/2} du$$

$$= 2\int_0^{\infty} e^{-u} u^{\frac{5}{2}-1} du = 2\Gamma\left(\frac{5}{2}\right)$$

$$= 2 \cdot \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 2 \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{3}{2}\sqrt{\pi} \quad \textbf{Answer} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 1.48. *Evaluate* $\int_0^{\infty} x^{n-1} e^{-h^2x^2} dx$.

Solution. We have $\quad l = \int_0^{\infty} x^{n-1} e^{-h^2x^2} dx$

Putting $h^2x^2 = y$, so that

$$h^2\, 2xdx = dy$$

$$\therefore \quad dx = \frac{1}{2}\frac{dy}{h^2x} = \frac{1}{2}\frac{dy}{h\sqrt{y}}$$

$$\therefore \quad I = \int_0^{\infty} e^{-y} \cdot \left(\frac{\sqrt{y}}{h}\right)^{n-1} \frac{1}{2}\frac{dy}{h\sqrt{y}}$$

$$= \frac{1}{2h^n}\int_0^{\infty} e^{-y} y^{\frac{n-2}{2}} dy$$

$$= \frac{1}{2h^n}\int_0^{\infty} e^{-y} y^{\frac{n}{2}-1} dy$$

$$= \frac{1}{2h^n}\Gamma\left(\frac{n}{2}\right) \quad \textbf{Answer} \qquad \text{[By definition of Gamma function]}$$

Example 1.49. *Evaluate* $\int_0^{\infty} x^6 e^{-2x} dx$.

Solution. We have $$I = \int_0^{\infty} e^{-2x}\ x^6 dx$$

Putting $2x = y \quad \Rightarrow \quad dx = \dfrac{dy}{2}$, then

$$I = \int_0^{\infty} e^{-y} \left(\frac{y}{2}\right)^6 \frac{dy}{2} = \frac{1}{2^7}\int_0^{\infty} e^{-y} y^{7-1} dy = \frac{1}{2^7}\Gamma(7)$$

$$I = \frac{1}{2^7}\Gamma(7) \qquad \text{[By definition]}$$

$$\Rightarrow \qquad = \frac{1}{2^7} 6! = \frac{45}{8} \quad \textbf{Answer} \qquad [\because \quad \Gamma(n+1) = n!, n > 0]$$

Example 1.50. *Prove that* $\int_0^{\infty} e^{(2ax - x^2)}\ dx = \frac{1}{2}\sqrt{\pi}\,.\,e^{a^2}$.

Solution. We have $$I = \int_0^{\infty} e^{(2ax - x^2)} dx = \int_0^{\infty} e^{a^2 - (x^2 - 2ax + a^2)}\ dx$$

$$= \int_0^{\infty} e^{a^2 - (x-a)^2}\ dx = e^{a^2}\int_0^{\infty} e^{-(x-a)^2} dx$$

Putting $(x - a) = y \quad \Rightarrow \quad dx = dy$

$$\therefore \qquad I = e^{a^2}\int_{-a}^{\infty} e^{-y^2}\ dy \qquad \text{...(1)}$$

Now by the definition of gamma function

$$\Gamma(n) = \int_0^{\infty} e^{-u} u^{n-1} du, n > 0$$

Put $n = \dfrac{1}{2}$, we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-u} u^{-1/2}\ du$$

Now put $u = y^2 \quad \Rightarrow \quad du = 2y\ dy$, we get

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-y^2} (y^2)^{-1/2}\,.\,2y dy \qquad \left[\because \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\Rightarrow \qquad \sqrt{\pi} = 2\int_0^{\infty} e^{-y^2} dy \qquad \text{...(2)}$$

$$\Rightarrow \qquad \int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

Using (2), (1) becomes $I = e^{a^2}\,.\,\dfrac{\sqrt{\pi}}{2} = \dfrac{1}{2}\sqrt{\pi}\,.\,e^{a^2}$ **Proved.**

Example 1.51. *Evaluate* $\int_0^{\infty} x^{-3/2}\,(1-e^{-x})\,dx$.

Solution. Let $$I = \int_0^{\infty} x^{-3/2}\,(1-e^{-x})\,dx$$

$$\Rightarrow \qquad I = \left[(1-e^{-x})\frac{x^{-1/2}}{\left(\frac{-1}{2}\right)}\right]_0^{\infty} - \int_0^{\infty} e^{-x}\cdot\frac{x^{-1/2}}{\left(\frac{-1}{2}\right)}\,dx$$

$$= 0 + 2\int_0^{\infty} e^{-x}x^{-1/2}\,dx$$

$$= 2\int_0^{\infty} e^{-x}x^{\frac{1}{2}-1}dx = 2\Gamma\left(\frac{1}{2}\right) \quad \textbf{Answer} \qquad \text{[By definition]}$$

$$= 2\sqrt{\pi}.$$

1.3.2 Beta Function

The Beta function is denoted and defined as

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad ...(1)$$

Where m, n are positive numbers, integer or fractional. This is also known as Eulerian integral of first kind.

1.3.2.1 Simple Properties of Beta Function

i. To show that: $B(m, n) = B(n, m)$ [Symmetry]

We have $$B(m, n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}\,[1-(1-x)]^{n-1}dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1}\,x^{n-1}dx = \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$= B(n, m)$$

Thus, $B(m, n) = B(n, m)$

ii. To show that: $\int_0^a x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1}\,B(m, n)$.

From (1), we have $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m, n)$

Putting $x = \dfrac{y}{a}$ $\Rightarrow$ $dx = \dfrac{dy}{a}$ and y varies from 0 to a.

$$\int_0^a \left(\frac{y}{a}\right)^{m-1}\left(1-\frac{y}{a}\right)^{n-1}\frac{dy}{a} = B(m, n)$$

$$\Rightarrow \qquad \frac{1}{a^{m+n-1}}\int_0^a y^{m-1}(a-y)^{n-1}dy = B(m, n)$$

$$\Rightarrow \qquad \int_0^a x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1}B(m, n)$$

iii. To show that $\quad B(m, n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

We know that $\quad B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Putting $x = \frac{1}{y+1}$ so that $dx = -\frac{dy}{(1+y)^2}$ and y varies from ∞ to 0.

$$\therefore \qquad B(m, n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left[1-\frac{1}{1+y}\right]^{n-1}\left[-\frac{dy}{(1+y)^2}\right]$$

$$\Rightarrow \qquad B(m, n) = \int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\cdot\frac{dy}{(1+y)^2}$$

$$B(m, n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \qquad \text{...(1)}$$

(In this integral m and n may be changed, by the virtue of symmetry of the function:)
Again, (1) may be written as

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \qquad \text{...(2)}$$

In order to solve second integral on R.H.S., put $y = \frac{1}{x}$, so that $dy = -\frac{1}{x^2}dx$ and x varies from 1 to 0.

$$\therefore \qquad \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy = \int_1^0 \left(\frac{1}{x}\right)^{n-1}\frac{x^{m+n}}{(1+x)^{m+n}}\left(-\frac{1}{x^2}\right)dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \text{...(3)}$$

Using (3), (2) becomes

$$B(m, n) = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(y)dy\right]$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

Hence, $\quad B(m, n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

iv. To show that $\quad B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

We know $\quad B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Putting $x = \sin^2\theta$ $\Rightarrow$ $dx = 2\sin\theta\cos\theta\, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \qquad B(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Thus $$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \,.\, \cos^{2n-1}\theta \,.\, d\theta$$

Particular case:

When $m = \frac{1}{2}$, $n = \frac{1}{2}$, above expression gives,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2 \,.\, \frac{\pi}{2} = \pi$$

v. Show that $(a-b)^{m+n-1} B(m, n) = \int_b^a (x-b)^{m-1} (a-x)^{n-1} dx$

We know that $$B(m, n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy$$

Let $y = \frac{x-b}{a-b}$ $\Rightarrow$ $dy = \frac{dx}{a-b}$ and y varies from b to a.

$$\therefore \qquad B(m, n) = \int_b^a \left(\frac{x-b}{a-b}\right)^{m-1} \left(1 - \frac{x-b}{a-b}\right)^{n-1} \frac{dx}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1} (a-x)^{n-1} dx$$

$$\therefore \qquad \int_b^a (x-b)^{m-1} (a-x)^{n-1}\, dx = (a-b)^{m+n-1} B(m, n)$$

Thus, $$\int_b^a (x-b)^{m-1} (a-x)^{n-1}\, dx = (a-b)^{m+n-1} B(m, n)$$

1.3.3 Relation between Beta and Gamma Function

To prove that $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m > 0$, $n > 0$

Proof: We know that

$$\frac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zy} y^{n-1} dy \qquad ...(1)$$

or, $$\Gamma(n) = \int_0^\infty z^n e^{-zy} y^{n-1} dy \qquad ...(2)$$

Also, $$\Gamma(m) = \int_0^\infty e^{-y} y^{m-1} dy \qquad ...(3)$$

Now multiply (2) by $z^{m-1} e^{-z}$ on both sides, we get

$$\Gamma(n) \,.\, e^{-z} \,.\, z^{m-1} = \int_0^{\infty} z^{m+n-1} e^{-zy} y^{n-1} e^{-z} dy$$

$$= \int_0^{\infty} z^{m+n-1} e^{-(y+1)z} y^{n-1}\, dy$$

Integrating both the sides w.r.t. z from 0 to ∞, we get

$$\Gamma(n) \int_0^{\infty} e^{-z} z^{m-1} dz = \int_0^{\infty}\left[\int_0^{\infty} z^{m+n-1} e^{-(y+1)z} dz\right] y^{n-1} dy$$

$$\Rightarrow \quad \Gamma(n)\,\Gamma(m) = \int_0^{\infty} \frac{\Gamma(m+n)}{(y+1)^{m+n}} \,.\, y^{n-1} dy \qquad \text{[by the property (1) and (3)]}$$

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n) \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \Gamma(m+n) \,.\, B(m, n) \qquad \text{[By (1) of } (iii) \text{ of 1.3.2]}$$

Thus, $$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},\ m > 0,\ n > 0$$

Deduction (*i*), $$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad \text{where } 0 < n < 1$$

Proof: We know that

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m, n) = \int_0^{x} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Choose $m + n = 1$, so that $m = (1 - n)$

$$\therefore \quad \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^{\infty} \frac{y^{n-1}}{1+y} dy,\ 0 < n < 1$$

$$\Rightarrow \quad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad \left[\because\ \Gamma(1) = 1, \int_0^{\infty} \frac{y^{n-1}}{1+y} dy = \frac{\pi}{\sin n\pi}\right]$$

where $0 < n < 1$

Deduction (*ii*) To prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: Since $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Put $n = \dfrac{1}{2}$, we obtain

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\Rightarrow \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Rightarrow \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Deduction (*iii*) To prove that

$$\int_0^{\pi/2} \cos^m\theta \sin^n\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}, \; m > -1, n > -1$$

Proof: Let $I = \int_0^{\pi/2} \cos^m\theta \sin^n\theta \, d\theta$

Put $\sin^2\theta = x$

$$\Rightarrow \qquad 2\sin\theta\cos\theta \, d\theta = dx$$

$$\Rightarrow \qquad d\theta = \frac{dx}{2\sqrt{x}\,.\sqrt{1-x}}$$

Also, when $\theta = 0$, $x = 0$ and when $\theta = \dfrac{\pi}{2}$, $x = 1$

$$\therefore \qquad I = \int_0^1 (1-x)^{\frac{m}{2}} \,.\, x^{\frac{n}{2}} \frac{dx}{2\sqrt{x}\,.\sqrt{(1-x)}}$$

$$= \frac{1}{2}\int_0^1 x^{\frac{n-1}{2}} \,.\, (1-x)^{\frac{m-1}{2}} \, dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{n+1}{2}-1} (1-x)^{\frac{m+1}{2}-1} \, dx$$

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{m+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)} \qquad \left[\because \; B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

SOME SOLVED EXAMPLES

Example 1.52. *Evaluate the integral* $\int_0^1 x^4 (1-\sqrt{x})^5 \, dx$.

Solution. Let $\sqrt{x} = y \;\Rightarrow\; x = y^2 \;\Rightarrow\; dx = 2y\,dy$

$\therefore$ given integral becomes

$$\int_0^1 y^8 (1-y)^5 \,.\, 2y\,dy = 2\int_0^1 y^{(10-1)} (1-y)^{6-1} \, dy$$

$$= 2B(10, 6) = 2\frac{\Gamma(10)\Gamma(6)}{\Gamma(16)} = \frac{2.9!5!}{15!} = \frac{1}{15015} \qquad \textbf{Answer}$$

Example 1.53. *Evaluate* $\int_0^1 (1-x^3)^{-1/2}\, dx$

Solution. Let $x^3 = y \;\Rightarrow\; x = y^{1/3} \;\Rightarrow\; dx = \frac{1}{3} y^{-2/3}\, dy$

$\therefore$ given integral becomes

$$\int_0^1 (1-x^3)^{-1/2}\, dx = \int_0^1 (1-y)^{-1/2} \cdot \frac{1}{3} y^{-2/3}\, dy$$

$$= \frac{1}{3}\int_0^1 y^{\frac{1}{3}-1} (1-y)^{\frac{1}{2}-1}\, dy$$

$$= \frac{1}{3} B\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)} \quad \textbf{Answer}$$

Example 1.54. *Express* $\int_0^1 y^m (1-y^p)^n\, dy$ *in terms of Beta function and hence evaluate* $\int_0^1 y^5 (1-y^3)^{10}\, dy$

Solution. Let $y^p = z \;\Rightarrow\; y = z^{1/p} \;\Rightarrow\; dy = \frac{1}{p} z^{\frac{1}{p}-1}\, dz$

$$\therefore \quad \int_0^1 y^m (1-y^p)^n\, dy = \int_0^1 z^{\frac{m}{p}} (1-z)^n \cdot \frac{1}{p} z^{\frac{1}{p}-1}\, dz$$

$$= \frac{1}{p}\int_0^1 z^{\frac{m+1}{p}-1} (1-z)^{\overline{n+1}-1}\, dz = \frac{1}{p} B\left(\frac{m+1}{p}, n+1\right) \qquad \ldots(1)$$

Putting $m = 5$, $p = 3$, $n = 10$ in (1), we obtain

$$\int_0^1 y^5 (1-y^3)^{10}\, dy = \frac{1}{3} B(2, 11)$$

$$= \frac{1}{3} \frac{\Gamma(2)\,\Gamma(11)}{\Gamma(13)} = \frac{1}{13} \frac{1 . \Gamma(1)\,\Gamma(11)}{12 . 11 . \Gamma(11)}$$

$$= \frac{1}{3 \times 12 \times 11} \qquad [\because \quad \Gamma(n+1) = n\,\Gamma(n)]$$

$$= \frac{1}{396} \quad \textbf{Answer}$$

Example 1.55. *Evaluate* $\int_0^\infty \frac{x^8 (1-x^6)}{(1+x)^{24}}\, dx$.

Solution. Let

$$I = \int_0^\infty \frac{x^8 (1-x^6)}{(1+x)^{24}}\, dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}\, dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\, dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\, dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\, dx$$

$$= B(9, 15) - B(15, 9)$$

$$= 0 \quad \textbf{Answer} \qquad [\because \quad B(m, n) = B(n, m)]$$

Example 1.56. *Prove that* $\int_0^{\pi/2}\left(1-\frac{\sin^2\theta}{2}\right)^{\frac{-1}{2}} d\theta = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\sqrt{\pi}}$.

Solution. Let
$$I = \int_0^{\pi/2}\left(1-\frac{\sin^2\theta}{2}\right)^{-1/2} d\theta = \sqrt{2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{2-\sin^2\theta}}$$
$$= \sqrt{2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1+\cos^2\theta}}$$

Put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt \Rightarrow d\theta = -\frac{dt}{\sqrt{1-t^2}}$

$\therefore$
$$I = -\sqrt{2}\int_1^0 \frac{dt}{\sqrt{1+t^2}\,.\,\sqrt{1-t^2}} = \sqrt{2}\int_0^1\frac{dt}{\sqrt{1-t^4}}$$

Again put $t^2 = \sin\theta \Rightarrow 2t\,dt = \cos\theta\, d\theta$

$\therefore$
$$I = \sqrt{2}\int_0^{\pi/2}\frac{\cos\theta\, d\theta}{2\sqrt{\sin\theta}\,.\,\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^0\theta\, d\theta$$
$$= \frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{2\sqrt{2}}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2\pi}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \qquad ...(1)$$

Since
$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \pi\sqrt{2}$$

$\therefore$ (1) gives,
$$I = \frac{1}{2\sqrt{2}}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2\sqrt{\pi}}{\pi\sqrt{2}} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{4\sqrt{\pi}} \qquad \textbf{Proved}$$

Example 1.57. *Evaluate* $\int_0^{\infty}\frac{xdx}{1+x^6}$.

Solution. Let
$$I = \int_0^{\infty}\frac{xdx}{1+x^6}$$

Putting $x^6 = y \Rightarrow x = y^{1/6} \Rightarrow dx = \frac{1}{6}y^{-5/6}\, dy$

$\therefore$
$$I = \int_0^{\infty}\frac{y^{1/6}}{(1+y)}\frac{1}{6}y^{-5/6}\, dy = \frac{1}{6}\int_0^{\infty}\frac{y^{-2/3}}{(1+y)}dy$$
$$= \frac{1}{6}\int_0^{\infty}\frac{y^{\frac{1}{3}-1}}{(1+y)^{\frac{1}{3}+\frac{2}{3}}}dy = \frac{1}{6}B\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{1}{6}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{6}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{3}} \qquad \left[\because\ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ and } \Gamma(1) = 1\right]$$

$$= \frac{\pi}{3\sqrt{3}}.$$

Example 1.58. *Prove that* $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$

Solution. We know that

$$\int_0^{\pi/2} \sin^n\theta \cos^0\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)} \qquad ...(1)$$

Now, let

$$I = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta\cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta\cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \qquad \text{[By (1)]}$$

$$= \frac{\pi\,\Gamma\left(\frac{1}{4}\right)}{4\,\Gamma\left(\frac{5}{4}\right)} = \frac{\pi\,\Gamma\left(\frac{1}{4}\right)}{4\,.\,\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi \quad \textbf{Proved} \qquad [\because\ \Gamma(n+1) = n\,\Gamma(n)]$$

Example 1.59. *Show that* $\int_0^{\infty} \frac{x^2 dx}{(1+x^4)^3} = \frac{5\pi\sqrt{2}}{128}.$

Solution. Put

$$x = \sqrt{\tan\theta} \quad\Rightarrow\quad dx = \frac{1}{2\sqrt{\tan\theta}}\,.\sec^2\theta\, d\theta$$

$$\therefore \qquad \int_0^{\infty} \frac{x^2 dx}{(1+x^4)^3} = \int_0^{\pi/2} \frac{\tan\theta\,.\,\frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta}{(1+\tan^2\theta)^3}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\tan\theta)^{1/2}(\sec\theta)^{-4}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\cos^{7/2}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)}$$

$$= \frac{1}{4.2!}\Gamma\left(\frac{3}{4}\right).\frac{5}{4}.\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{5}{128}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{5}{128}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{5\pi\sqrt{2}}{128} \qquad \textbf{Proved}$$

Example 1.60. *Evaluate* $\int_0^\infty e^{-x^{1/3}}\,dx$.

Solution. Let $x^{1/3} = y \quad \Rightarrow \quad x = y^3 \quad \Rightarrow \quad dx = 3y^2\,dy$

$$\therefore \qquad \int_0^\infty e^{-x^{1/3}}dx = \int_0^\infty e^{-y}\,.\,3y^2dy$$

$$= 3\int_0^\infty e^{-y}y^{3-1}dy = 3\Gamma(3) = 3\,.\,2\,! = 6 \qquad \textbf{Answer}$$

1.3.4 Duplication Formula

To show that $\Gamma(p)\,\Gamma\left(p+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2p-1}}\,\Gamma(2p)$ where $p > 0$

Proof: We know that

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \qquad ...(1)$$

Now putting $n = m$ in (1), we obtain

$$\frac{\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{2\Gamma(m+1)} = \int_0^{\pi/2}(\sin\theta\cos\theta)^m\,d\theta$$

$$= \frac{1}{2^m}\int_0^{\pi/2}(2\sin\theta\cos\theta)^m\,d\theta = \frac{1}{2^m}\int_0^{\pi/2}(\sin 2\theta)^m\,d\theta$$

Again putting $2\theta = \phi \quad \Rightarrow \quad d\theta = \frac{d\phi}{2}$, we get

$$\frac{\left[\Gamma\left(\frac{m+1}{2}\right)\right]^2}{2\Gamma(m+1)} = \frac{1}{2^m}\int_0^\pi \sin^m\phi\,.\,\frac{d\phi}{2}$$

$$= \frac{1}{2^m}\int_0^{\pi/2}\sin^m\phi\cos^0\phi\,d\phi \qquad \left\{\int_0^{2a}f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)\right\}$$

$$= \frac{1}{2^m}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{m+2}{2}\right)} \qquad \text{[by (1)]}$$

$$\Rightarrow \qquad \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1)} = \frac{\sqrt{\pi}}{2^m} \frac{1}{\Gamma\left(\frac{m+2}{2}\right)}$$

Let $\quad \dfrac{m+1}{2} = p \;\Rightarrow\; m = 2p - 1, p > 0$

$$\therefore \qquad \frac{\Gamma(p)}{\Gamma(2p)} = \frac{\sqrt{\pi}}{2^{2p-1}} \cdot \frac{1}{\Gamma\left(\frac{2p+1}{2}\right)}$$

$$\Rightarrow \qquad \Gamma(p)\,\Gamma\left(p+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2p-1}}\,.\,\Gamma(2p) \text{ where } p > 0$$

which is known as Duplication formula

Deduction (*i*) To prove that

$$2^m\,\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+2}{2}\right) = \sqrt{\pi}\;\Gamma(m+1)$$

for all real values of m

Proof: Put $2p - 1 = m$ in Duplication formula, we get

$$\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+2}{2}\right) = \frac{\sqrt{\pi}}{2^m}\Gamma(m+1)$$

$$\Rightarrow \qquad 2^m\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+2}{2}\right) = \sqrt{\pi}\,\Gamma(m+1) \qquad \textbf{(proved)}$$

Deduction (*ii*) To prove that

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m)!}{2^m\,.\,m!}\sqrt{\pi}\text{, where } m > 0$$

Proof: We have $\quad \dfrac{\Gamma(2m)}{\Gamma(m)} = \dfrac{(2m-1)!}{(m-1)!} = \dfrac{2m\,.\,(2m-1)!}{2m\,.\,(m-1)!} = \dfrac{(2m)!}{2\,.\,m!}$ ...(1)

Now using Duplication formula, we have

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\frac{\Gamma(2m)}{\Gamma(m)}$$

$$= \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\frac{(2m)!}{2\,.\,(m)!} \qquad \text{[by (1)]}$$

$$= \frac{\sqrt{\pi}}{2^m}\,\frac{(2m)!}{m!} \qquad \textbf{Proved}$$

SOME SOLVED EXAMPLES

Example 1.61. *Prove that* $B(n, n) = 2^{1-2n}\,B\left(n, \frac{1}{2}\right)$.

Solution. We have $B\left(n, \frac{1}{2}\right) = \dfrac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} = \dfrac{\Gamma(n)\,\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)}$

$$= \frac{\Gamma(n)\,\Gamma(n)\,\sqrt{\pi}}{\sqrt{\pi}\,\Gamma(2n).2^{1-2n}} \qquad \text{[By Duplication formula]}$$

$$= \frac{B(n, n)}{2^{1-2n}}$$

$$\Rightarrow \qquad B(n, n) = 2^{1-2n}\, B\left(n, \frac{1}{2}\right)$$

Example 1.62. *Prove that* $\Gamma\left(\frac{1}{6}\right) = \dfrac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{3}\right)\right]^2$.

Solution. By Duplication formula, we have

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$$

Putting $m = \frac{1}{6}$, we get

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{2^{-2/3}}$$

$$\therefore \qquad \Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{2^{-2/3}\,\Gamma\left(\frac{2}{3}\right)} \qquad \text{...(1)}$$

Again, we know that

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Putting $n = \frac{1}{3}$, we obtain

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

$$\therefore \qquad \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)} \qquad \text{...(2)}$$

Using (2), (1) becomes

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{2^{-2/3}} . \frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)}{2\pi} = \frac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{3}\right)\right]^2$$ **Proved**

Example 1.63. *Show that* $\int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}x^2\right) dx = 1$ *using Beta-Gamma functions.*

Solution. Let
$$I = \int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}x^2\right) dx = 2\int_0^{\infty} \cos\left(\frac{\pi}{2}x^2\right) dx \qquad ...(1)$$

Now substituting $\frac{\pi x^2}{2} = y \quad \Rightarrow \quad x = \sqrt{\frac{2}{\pi}}\, y^{1/2}$

$$\therefore \qquad dx = \frac{1}{2}\left(\sqrt{\frac{2}{\pi}}\right) y^{-1/2}\, dy$$

$$I = 2\,.\frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} \cos y\, dy = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} \cos y\, dy$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} e^{-iy}\, dy \qquad [\because \; e^{-i\theta} = \cos\theta - i\sin\theta]$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{-1/2} e^{-iy}\, dy$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}}\, \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{e^{i\pi/2}}} \qquad \left[\because \; \int_0^{\infty} e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \text{R.P. of } \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{\pi}}{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}} \qquad [\because \; i = e^{i\pi/2}]$$

$$= \text{R.P. of } \sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-1/2}$$

$$= \text{R.P. of } \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) \qquad \text{[by Demoivre's theorem]}$$

$$= \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\,.\frac{1}{\sqrt{2}} = 1 \quad \textbf{Proved}$$

Example 1.64. *Calculate* $\int_0^{\infty} \cos(\lambda^2 x^2)\, dx$, *using Beta-Gamma functions.*

Solution. Let
$$I = \int_0^{\infty} \cos(\lambda^2 x^2)\, dx$$

Putting $x^2 = z$

$$\Rightarrow \qquad dx = \frac{1}{2} z^{\frac{-1}{2}}\, dz$$

$$\therefore \qquad I = \int_0^{\infty} \cos\lambda^2 z\,.\frac{1}{2} z^{\frac{-1}{2}}\, dz = \frac{1}{2}\int_0^{\infty} z^{\frac{1}{2}-1} \cos\lambda^2 z\, dz$$

$$= \text{R.P. of } \frac{1}{2}\int_0^\infty e^{i\lambda^2 z} z^{\frac{1}{2}-1}\, dz$$

$$= \text{R.P. of } \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{(i\lambda^2)^{1/2}}$$

$$= \text{R.P. of } \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{\lambda(e^{i\pi/2})^{1/2}} \qquad [(i) = e^{i\pi/2}]$$

$$= \text{R.P. of } \frac{\sqrt{\pi}}{2\lambda}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-1/2} \qquad \left[\because \int_0^\infty e^{-kx} . x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}\right]$$

$$= \text{R.P. of } \frac{\sqrt{\pi}}{2\lambda}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) \qquad [\because \text{ by Demoivre's theorem}]$$

$$= \frac{\sqrt{\pi}}{2\lambda}\cos\frac{\pi}{4} = \frac{1}{2\lambda}\sqrt{\frac{\pi}{2}} \qquad \textbf{Proved}$$

Example 1.65. *Evaluate* $\int_0^1 \log \Gamma(y)\, dy$ *, using Beta-Gamma functions.*

Solution. Let $I = \int_0^1 \log \Gamma(y)\, dy$ ...(1)

or, $I = \int_0^1 \log \Gamma(1-y)\, dy$ ...(2) $\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

Adding (1) and (2), we get

$$2I = \int_0^1 [\log\Gamma(y) + \log\Gamma(1-y)]\, dy$$

$$= \int_0^1 \log[\Gamma(y)\,\Gamma(1-y)]\, dy$$

$$= \int_0^1 \log\left(\frac{\pi}{\sin \pi y}\right) dy \qquad \left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \int_0^1 (\log\pi - \log\sin\pi y)\, dy$$

$$= \log\pi\,[y]_0^1 - \int_0^\pi \log\sin z . \frac{1}{\pi}\, dz \qquad [\text{for } 2^{\text{nd}} \text{ integral, put } \pi y = z]$$

$$= \log\pi - \frac{2}{\pi}\int_0^{\pi/2} \log\sin z\, dz \qquad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(a-x) = f(x)\right]$$

$$= \log\pi - \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right) \qquad \left[\because \int_0^{\pi/2} \log\sin x\, dx = -\frac{\pi}{2}\log 2\right]$$

$$= \log\pi + \log 2 = \log 2\pi \qquad [\text{Refer Example 1.31}]$$

$$\therefore \qquad I = \frac{1}{2}\log 2\pi \qquad \textbf{Answer}$$

Example 1.66. *Prove that* $\Gamma\left(\frac{3}{2}-m\right)\Gamma\left(\frac{3}{2}+m\right)=\left(\frac{1}{4}-m^3\right)$ *π sec πm, provided* $-1 < 2m < 1$.

Solution.

$$\text{L.H.S.} = \Gamma\left(\frac{3}{2}-m\right)\Gamma\left(\frac{3}{2}+m\right)$$

$$= \left(\frac{1}{2}-m\right)\Gamma\left(\frac{1}{2}-m\right).\left(\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}+m\right)$$

$$= \left(\frac{1}{4}-m^2\right)\Gamma\left(\frac{1}{2}-m\right)\Gamma\left(\frac{1}{2}+m\right) = \left(\frac{1}{4}-m^2\right)\Gamma\left(\frac{1-2m}{2}\right)\Gamma\left(1-\frac{1-2m}{2}\right)$$

$$= \left(\frac{1}{4}-m^2\right)\frac{\pi}{\sin\left(\frac{1-2m}{2}\right)\pi} = \left(\frac{1}{4}-m^2\right)\frac{\pi}{\sin\left(\frac{\pi}{2}-\pi m\right)}$$

$$= \left(\frac{1}{4}-m^2\right)\frac{\pi}{\cos \pi m} = \left(\frac{1}{4}-m^2\right)\pi \sec \pi m$$

= R.H.S. **Proved**

Example 1.67. *Show that* $\int_0^{\pi/2} \tan^n\theta\, d\theta = \frac{\pi}{2} \sec\frac{n\pi}{2}$, $-1 < n < 1$.

Solution. Let

$$I = \int_0^{\pi/2} \tan^n \theta\, d\theta = \int_0^{\pi/2} \sin^n \cos^{-n} \theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)$$

$$= \frac{1}{2}\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} = \frac{1}{2}\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{1}{2}\frac{\pi}{\cos\frac{n\pi}{2}}$$

$$= \frac{1}{2}\sec\frac{n\pi}{2}$$ **Proved**

EXERCISE 1.6

1. Evaluate the following integrals:

a. $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$

b. $\int_0^\infty (8-x^3)^{-1/3}\, dx$

c. $\int_0^\infty e^{-x^2} x^{-1/2} dx \int_0^\infty x^2 e^{-x^4}\, dx$

d. $\int_0^\infty x^6 e^{-2x} dx$

e. $\int_0^\infty \frac{e^{-pt}}{\sqrt{t}}\, dt$

f. $\int_0^\infty \frac{x\, dx}{1+x^6}$

g. $\int_0^3 \frac{dt}{\sqrt{3t-t^2}}$

h. $\int_0^1 \frac{dt}{\sqrt{-\log t}}$

2. Prove that $\int_0^1 t^m (\log t)^n \, dt = \dfrac{(-1)^n \, . \, n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

3. Show that $\int_0^\infty \dfrac{t^{m-1}}{(a+bt)^{m+n}} dt = \dfrac{B(m,n)}{a^n b^m}$, where m, n, a and b are positive integers.

4. Prove that $\int_0^\pi \sqrt{\dfrac{\sin\theta}{(5+3\cos\theta)^n}} \, d\theta = \dfrac{\left[\Gamma\left(\frac{3}{4}\right)\right]^2}{2\sqrt{2\pi}}$

5. Evaluate the integrals:

i. $\int_0^1 \dfrac{dx}{(1-x^n)^{1/n}}$ ii. $\int_0^1 x^m (1-x^m)^p \, dx$ iii. $\int_0^1 \dfrac{dt}{(1-t^n)^{1/2}}$ iv. $\int_0^\infty x^m e^{-ax^n} dx$

6. Evaluate i. $\Gamma\left(-\dfrac{5}{2}\right)$ ii. $\Gamma\left(-\dfrac{15}{2}\right)$

7. Evaluate the following integrals:

i. $\int_0^1 \sqrt{1-x^4} \, dx$ ii. $\int_0^1 \dfrac{x^{m-1}}{(1+ax)(1-x)^m} dx$ iii. $\int_0^{\pi/2} \sin^2 x \, dx \times \int_0^{\pi/2} \sin^{q+1} x \, dx$

8. Show that $1 \, . \, 3 \, . \, 5 \, . \dots (2m-1) = \dfrac{2^m \Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}}$

9. Prove that $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta = \dfrac{1}{2}\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right)$

10. Prove that $\int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\sec\theta}) \, d\theta = \dfrac{1}{2}\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right) + \sqrt{\pi}\,\Gamma\left(\dfrac{3}{4}\right)$

Answers

1. a. $\dfrac{\sqrt{\pi}}{3}$ b. $\dfrac{2\pi}{3\sqrt{3}}$ c. $\dfrac{\pi}{4\sqrt{2}}$ d. $\dfrac{45}{8}$

e. $\sqrt{\dfrac{\pi}{p}}, \, p > 0$ f. $\dfrac{\pi}{3\sqrt{3}}$ g. π h. $\sqrt{\pi}$

5. i. $\dfrac{\pi}{n} \sin\dfrac{\pi}{n}$ ii. $\dfrac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{n\Gamma\left(p+1+\frac{m+1}{n}\right)}$ iii. $\dfrac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{\pi\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$ iv. $\dfrac{1}{n\,a^{(m+1)/n}}\Gamma\left(\dfrac{m+1}{n}\right)$

6. i. $\dfrac{-8}{15}\sqrt{\pi}$ ii. $\dfrac{\sqrt{\pi}2^8}{1.3.5.7\dots15}$

7. i. $\dfrac{[\Gamma(1/4)^2]}{6\sqrt{2\pi}}$ ii. $\dfrac{1}{(1+a)^m} \cdot \dfrac{\pi}{\sin m\pi}$ iii. $\dfrac{\pi}{2(q+1)}$

INTERESTING FACT

From Feynman diagrams (which involves the pictorial representation of subatomic particles), to Maxwell-Boltzmann statistics and distribution (which used in physics, chemistry and statistical mechanics to determine speed of molecules), these functions includes some real ground applications.

VIDEO REFERENCES

Beta & Gamma Function

USES OF ICT

- https://youtu.be/9_m36W3cK74
- https://youtu.be/3Co68ALYRTI

APPLICATIONS TO REAL LIFE

- It has many applications in strong nuclear forces.
- Beta distribution is used, when we solve time management problems.
- Gamma function is used to find time-based occurrences, such as life span of anything.
- In packing problems like, a cube will fit better in a sphere or a sphere in a cube.

1.4 APPLICATIONS OF DEFINITE INTEGRALS TO EVALUATE SURFACE AREAS AND VOLUMES OF REVOLUTION

If a plane area is revolved about a fixed line in its own plane, then the body so generated by the plane area is called the volume of the solid of revolution and the surface so generated is called the surface of revolution and the fixed line about which the solid revolves is called the axis of revolution.

Examples:

1. When a right angled triangle is revolved about its hypotenuse, then the double cone is formed.
2. When a circle is rotated about its diameter, a sphere is generated.

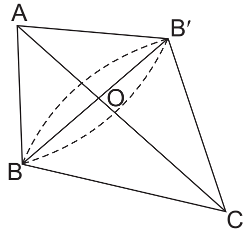


Fig. 1.17

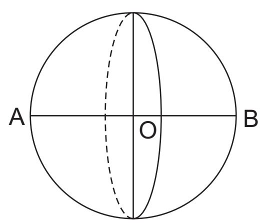


Fig. 1.18

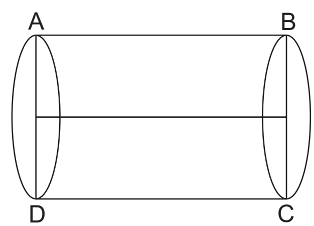


Fig. 1.19

3. When a square is rotated about any of its side, a right circular cylinder generated.

1.4.1 Volumes of Solids of Revolution

1.4.1.1 For Cartesian Curves

1. **Revolution about x-axis:** The volume of solid generated by the revolution about x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates, $x = a$, $x = b$ is $\int_a^b \pi y^2 dx$.
2. **Revolution about y-axis:** The volume of solid generated by the revolution about y-axis of the area bounded by the curve $x = f(y)$, the y-axis and the abscissas $y = a$, $y = b$ is $\int_a^b \pi y^2\, dx$.
3. **Revolution about any axis:** The volume of the solid generated by the revolution about any axis CD of the area bounded by the curve AB, the axis CD and the perpendiculars AC, BD on the axis is $\int_{OC}^{OD} \pi(PM)^2\, d(OM)$ where O is a fixed point on the axis CD and PM is perpendicular from any point P of the curve AB on CD.

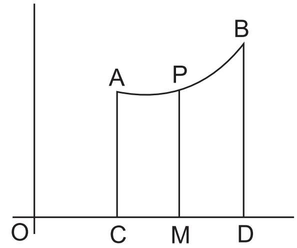


Fig. 1.20

1.4.1.2 For Parametric Curves

1. The volume of the solid generated by the revolution about x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points $t = a$, $t = b$ is $\int_a^b \pi x^2 \frac{dy}{dt} dt$.
2. The volume of the solid generated by the revolution about y-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the y-axis and the abscissas at the points $t = a$, $t = b$ is $\int_a^b \pi x^2 \frac{dy}{dt} dt$.
3. Volume of solid of revolution between two solids: The volume of the solid generated by the revolution about the x-axis of the area bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$, $x = b$ is $\int_a^b \pi[f_1^2(x) - f_2^2(x)]\,dx$ where $f_1(x)$ is the ordinates of the upper curve and $f_2(x)$ is that of the lower curve.

1.4.1.3 For Polar Curves

The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$

1. about the initial line OX $(\theta = 0)$ is $\int_\alpha^\beta \frac{2}{3}\pi r^3 \sin\theta\, d\theta$
2. about the line OY $\left(\theta = \frac{\pi}{2}\right)$ is $\int_\alpha^\beta \frac{2}{3}\pi r^3 \cos\theta\, d\theta$

1.4.2 Surface Areas of Solid of Revolution

1.4.2.1 For Cartesian Curve

The curved surface of the solid generated by the revolution about x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b 2\pi y \frac{ds}{dx} dx$ where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

1.4.2.2 For Parametric Curve

The curved surface of the solid generated by the revolution about x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points $t = a$, $t = b$ is

$$\int_a^b 2\pi y \frac{ds}{dt}\,dt \quad \text{where} \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

1.4.2.3 For Polar Curve

The curved surface of the solid generated by the revolution, about the initial line of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi y \frac{ds}{d\theta}\,d\theta \quad \text{where} \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad \text{and } y = r\sin\theta.$$

SOME SOLVED EXAMPLES

Example 1.68. *Find the volume of the solid generated by revolving the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *about the major axis.*

Solution. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

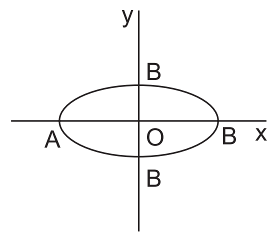


Fig. 1.21

The ellipse is symmetrical about y-axis.

Required volume of solid generated by the ellipse about x-axis

$$= 2\int_0^a \pi y^2\,dx = 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= 2\pi \frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx = 2\pi \frac{b^2}{a^2}\left|a^2 x - \frac{x^3}{3}\right|_0^a = \frac{4}{3}\pi ab^2$$

Example 1.69. *Find the volume generated by the revolution about the initial line of the limacon* $r = a + b\cos\theta$, $a > b$.

Solution. We shall revolve only the shaded region above the initial line

$$V = \frac{2}{3}\pi \int_0^{\pi} (a + b\cos\theta)^2 \sin\theta\,d\theta$$

$$a + b\cos\theta = t \quad b\sin\theta\,d\theta = -\,dt$$

$$= -\frac{2}{3}\frac{\pi}{b}\left[\frac{t^3}{3}\right]_{a+b}^{a-b} = -\frac{2\pi}{3b}\left[\frac{-6a^2b - 2b^3}{3}\right]$$

$$= \frac{4}{9}\pi(b^2 + 3a^2).$$

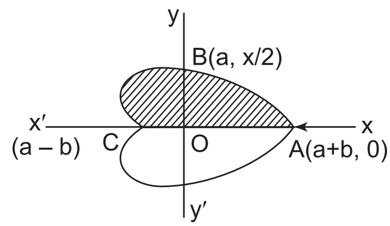


Fig. 1.22

Example 1.70. *Find the volume of the solid generated by the revolution of the loop of the curve $(a-x)\,y^2 = (a+x)\,x^2$ about the x-axis.*

Solution. The shape of the curve is shown in the figure.

The curve is symmetrical about x-axis

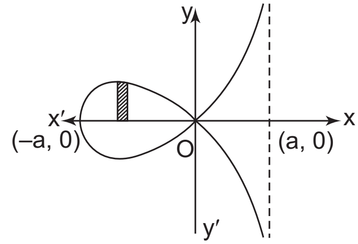


Fig. 1.23

$$\therefore \quad \text{Required volume}, V = \int_{-a}^{0} \pi y^2 dx = \int_{-a}^{0} \pi \frac{(a+x)}{(a-x)} x^2 dx$$

Let $\quad a - x = z,\ dx = -\,dz$

when $\quad x = -a,\ z = 2a$

$\quad x = 0,\ \ z = a$

$$V = \pi \int_{2a}^{a} \frac{(a+a-z)}{z} (a-z)^2\,(-dz) = \pi \int_{a}^{2a} \frac{2a-z}{z} (a-z)^2\, dz$$

$$= \pi \int_{a}^{2a} \frac{(2a-z)}{z} (a^2 + z^2 - 2z)\, dz$$

$$= \pi \int_{a}^{2a} \left[\frac{2a^3}{z} + 2az - 4a^2 - a^2 - z^2 + 2az \right] dz$$

$$= \pi \int_{a}^{2a} \left[\frac{2a^3}{z} + 4az - z^2 - 5a^2 \right] dz \ = \pi \left[2a^3 \log z + 2az^2 - \frac{z^3}{3} - 5a^2 z \right]_{a}^{2a}$$

$$= \pi \left[\left\{ 2a^3 \log 2a + 2a\,(2a)^2 - \frac{(2a)^3}{3} - 5a^2\,(2a) \right\} - \left\{ 2a^3 \log a + 2a^3 - \frac{a^3}{3} - 5a^3 \right\} \right]$$

$$= 2\pi a^3 \left[\log 2 - \frac{2}{3} \right]$$

Example 1.71. *Find the volume generated by the revolution of the area under one complete arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, the axis of revolution being the x-axis.*

Solution. Let the area under arc be divided into n strips of width dx by lines || to the y-axis and y be the height of the typical strip at a distance x from y-axis.

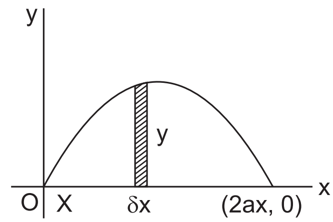


Fig. 1.24

$$\text{Volume} = \int_0^{2\pi} \pi y^2 dx = \int_0^{2\pi} \pi a^2\,(1-\cos\theta)^2\ a\,(1-\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} \pi a^3\,(1-\cos\theta)^3\ d\theta$$

$$= \int_0^{2\pi} \pi a^3 \times \left(2\sin^2 \frac{\theta}{2} \right)^3 d\theta$$

$$= 8\pi a^3 \int_0^{2\pi} \sin^6 \left(\frac{\theta}{2} \right) d\theta = 16\pi a^3 \int_0^{\pi} \sin^6 \phi\, d\phi \qquad (\theta = 2\phi)$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6 \phi\, d\phi \qquad (\sin(\pi - \phi) = \sin\phi)$$

$$= 32\pi a^3\ \frac{5\times 3\times 1}{6\times 4\times 2}\,\frac{\pi}{2} = 5\pi^2 a^3. \qquad (\because \text{ By using Reduction formula})$$

Example 1.72. *Find the volume of the surface generated by revolving the Cardioid r = a(1 + cos θ) about initial line.*

Solution. Volume generated

$$= \frac{2}{3}\pi\int_0^{\pi} r^3 \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2}{3}\pi a^3 \int_0^{\pi}(1+\cos\theta)^3 (-\sin\theta)\, d\theta$$

$$= \frac{-2\pi}{3} a^3 \left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi} = \frac{2}{3}\pi a^3 \left[\frac{(1+1)^4}{4}\right] = \frac{8}{3}\pi a^3$$

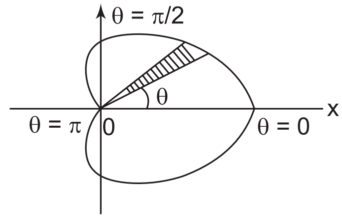


Fig. 1.25

Example 1.73. *Show that the volume of the solid formed by the revolution of the Cissoid $y^2(2a - x) = x^3$ about its asymptotes is $2\pi^2a^3$.*

Solution. The asymptote of this curve is $x = 2a$.

Required volume $\quad V = 2\pi\int_0^{2a} (2a-x)^2\, dy$

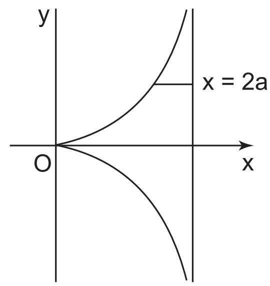


Fig. 1.26

From the equation of the curve

$$y = \frac{x^{3/2}}{\sqrt{2a-x}}$$

$$\frac{dy}{dx} = \frac{(3a-x)\sqrt{x}}{(2a-x)^{3/2}}$$

$$dy = \frac{(3a-x)\sqrt{x}}{(2a-x)^{3/2}}\, dx$$

When y varies from 0 to ∞, x varies from 0 to $2a$

$$V = 2\pi\int_0^{2a}(2a-x)^2\, dy = 2\pi\int_0^{2a}(2a-x)^2 \frac{(3a-x)\sqrt{x}}{(2a-x)^{3/2}}\, dx$$

$$= 2\pi\int_0^{2a}(a-x)\sqrt{2ax-x^2}\, dx + 4\pi a\int_0^{2a}\sqrt{2ax-x^2}\, dx$$

Put $x = 2a\sin^2\theta$, we get

$$\int_0^{2a}\sqrt{2ax-x^2}\, dx = \int_0^{\pi/2} 2a\times 4a\sin^2\theta\cos^2\theta\, d\theta = \pi\frac{a^2}{2}$$

$$V = \pi\left[\frac{(2ax-x^2)^{3/2}}{3/2}\right]_0^{2a} + 4\pi a\frac{\pi a^2}{2} = 0 + 2\pi^2 a^3 = 2\pi^2 a^3$$

Example 1.74. *The area cut off from the right parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum through four right-angles about the chord. Find the volume of the solid generated.*

Solution. The equation of the parabola is

$$y^2 = 4ax$$

The co-ordinates of B are $(a, 2a)$

Hence the equation of OB

$$y - 0 = \frac{2a - 0}{a - 0}(x - 0)$$

$$2x - y = 0$$

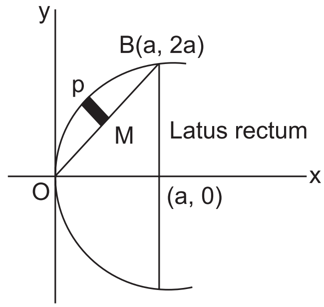


Fig. 1.27

Let $P(at^2, 2at)$ be a point on the arc OB and PM the perpendicular from P to OB.

$$PM = \frac{2at^2 - 2at}{\sqrt{4+1}} = \frac{2at\,(t-1)}{\sqrt{5}}$$

$$OP = \sqrt{(at^2 - 0)^2 + (2at - 0)^2} = at\sqrt{t^2 + 4}$$

$$OM^2 = OP^2 - PM^2 = a^2 t^2(t^2 + 4) - \frac{4a^2t^2\,(t-1)^2}{5}$$

$$OM^2 = \frac{a^2t^2(t+4)^2}{5}$$

$$OM = \frac{at\,(t+4)}{\sqrt{5}}$$

$$\text{Required volume} = \int_{t=0}^{t} \pi(PM)^2\, d(OM) = \pi\int_0^1 \frac{4a^2t^2(t-1)^2}{5}\, d\left[\frac{at\,(t+4)}{\sqrt{5}}\right]$$

$$= \frac{4\pi a^3}{5\sqrt{5}}\int_0^1 t^2(t^2 - 2t + 1)(2t + 4)\,dt = \frac{4\pi a^3}{5\sqrt{5}}\int_0^1 (2t^5 - 6t^3 + 4t^2)dt = \frac{2\pi a^3}{15\sqrt{5}}$$

Example 1.75. *Find the surface area of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about the major axis.*

Solution. The equation of the ellipse is

$$x^2 + 4y^2 = 16$$

$$4y^2 = 16 - x^2 \quad \therefore \quad y = \frac{\sqrt{16 - x^2}}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{2}\frac{1}{\sqrt{16 - x^2}} \times (-2x) = -\frac{x}{2\sqrt{16 - x^2}}$$

$$dy = -\frac{x}{2\sqrt{16 - x^2}}\,dx$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{4(16 - x^2)}} = \sqrt{\frac{64 - 3x^2}{4(16 - x^2)}}$$

The ellipse $x^2 + 4y^2 = 16$ meets x-axis where $y = 0$,

$$x^2 = 16 \quad x = \pm 4$$

For the upper half of the ellipse in first quadrant x varies from 0 to 4.

The ellipse is symmetrical about y-axis

$$\therefore \quad \text{Required surface} = 2 \times \int_0^4 2\pi\, y\,\frac{ds}{dx}\,dx = 4\pi\int_0^4 \frac{\sqrt{16 - x^2}}{2} \times \sqrt{\frac{64 - 3x^2}{4(16 - x^2)}}\,dx$$

$$= \pi \int_0^4 \sqrt{64 - 3x^2}\, dx = \sqrt{3}\,\pi \int_0^4 \sqrt{\frac{64}{3} - x^2}\, dx$$

$$= \sqrt{3}\pi \left[\frac{x\sqrt{\frac{64}{3} - x^2}}{2} + \frac{64}{3 \times 2} \sin^{-1} \frac{x}{8} \right]_0^4$$

$$= \sqrt{3}\,\pi \left[2\sqrt{\frac{64}{3} - 16} + \frac{32}{3} \sin^{-1} \frac{\sqrt{3}}{2} \right]$$

$$= \sqrt{3}\,\pi \left[2 \times \frac{4}{\sqrt{3}} + \frac{32}{3}\,\frac{\pi}{3} \right] = 8\pi \left[1 + \frac{4\pi}{3\sqrt{3}} \right]$$

Example 1.76. *Find the area of the surface of revolution generated by revolving one arc of the curve $y = \sin x$ about the x-axis.*

Solution. One arc of the curve $y = \sin x$ lies in $(0, \pi)$. Further

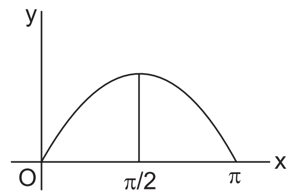


Fig. 1.28

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \cos^2 x}$$

$$\text{Required surface area} = 2\pi \int_0^\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x}\, dx \qquad \text{Put } t = \cos x \quad \therefore \quad dt = -\sin x\, dx$$

$$= -2\pi \int_1^{-1} \sqrt{1 + t^2}\, dt = 2\pi \int_{-1}^1 \sqrt{1 + t^2}\, dt = 2\pi \left[\frac{t\sqrt{1 + t^2}}{2} + \frac{1}{2} \sin h^{-1}(1) \right]_{-1}^1$$

$$= \pi \{ [\sqrt{2} + \sinh^{-1}(1)] - [-\sqrt{2} - \sinh^{-1}(1)] \} = 2\pi [\sqrt{2} + \sinh^{-1}(1)]$$

Example 1.77. *Find the surface of the solid generated by revolving the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ about the axis of x.*

Solution. Let the shaded portion OAB be revolved above x-axis.

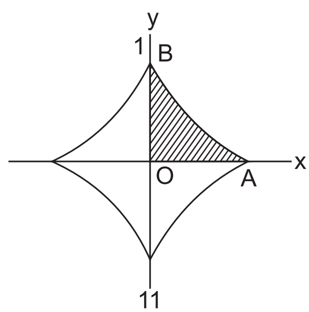


Fig. 1.29

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-3a\cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2}$$

$$= \sqrt{9a^2 \cos^2 t \sin^2 t \times (\cos^2 t + \sin^2 t)}$$

$$= 3a \cos t \sin t.$$

$$\text{Surface area,} \quad S = 2\int_0^{\pi/2} 2\pi\, y \frac{ds}{dt} dt = 4\pi \int_0^{\pi/2} a \sin^3 t \times 3a \cos t \sin t\, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t\, dt = 12\pi a^2 \left[\frac{\sin^5 t}{5} \right]_0^{\pi/2} = \frac{12}{5}\pi a^2$$

Example 1.78. *A quadrant of a circle of radius 'a' bounded by the tangents at its extremities revolves about one of the tangents. Show that the surfaces area so generated is $\pi(\pi - 2)a^2$.*

Solution. Let the equation of the circle be

$$x^2 + y^2 = a^2 \text{ or } x = a\cos t,\ y = a\sin t$$

and $P(x, y)$ be any point on it

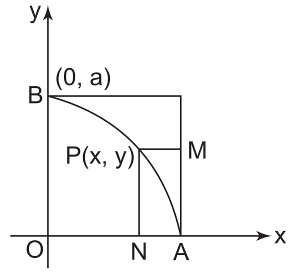


Fig. 1.30

$$PM = NA = a - x = a - a\cos t = a(1 - \cos t)$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

Surface area, $$S = \int_0^{\pi/2} 2\pi\,(PM)^2 \times \frac{ds}{dt} \times dt$$

$$= 2\pi a^2 \int_0^{\pi/2} (1 - \cos t)\,dt$$

$$= 2\pi a^2 \int_0^{\pi/2} (1 - \cos t)\,dt = 2\pi a^2\,[t - \sin t]_0^{\pi/2}$$

$$= 2\pi a^2 \left[\frac{\pi}{2} - 1\right]$$

$$= \pi a^2(\pi - 2)$$

Example 1.79. *Find the surface of the solid formed by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.*

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π.

Also $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\,(1 + \cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1 + \cos\theta)} = a\sqrt{4\cos^2\theta/2} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface $$= \int_0^{\pi} 2\pi\,y\,\frac{ds}{d\theta}\,d\theta = 2\pi\int_0^{\pi} r\sin\theta \times 2a\cos\frac{\theta}{2}\,d\theta \qquad \left\{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right.$$

$$= 4\pi a\int_0^{\pi} a(1 + \cos\theta)\sin\theta\cos\frac{\theta}{2}\,d\theta$$

$$= 4\pi a^2\int_0^{\pi} 2\cos^2\frac{\theta}{2} \times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\,(-2)\int_0^{\pi}\cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2} \times \frac{1}{2}\right)d\theta$$

$$= -32\pi a^2\left[\frac{\cos^5\theta/2}{5}\right]_0^{\pi} = -\frac{32\pi a^2}{5}(0 - 1) = \frac{32\pi a^2}{5}$$

Example 1.80. *Find the surface of the sphere of radius 'a', equation of the circle being $r = a$.*

Solution. Equation of the circle is $r = a$. Let the shaded portion be revolved about x-axis.

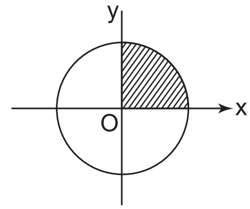


Fig. 1.31

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 + 0} = a$$

$$S = 2\int_0^{\pi/2} 2\pi r \sin\theta \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi/2} a \sin\theta \times a\, d\theta$$

$$= 4\pi \int_0^{\pi/2} a^2 \sin\theta\, d\theta$$

$$= 4\pi a^2 [-\cos\theta]_0^{\pi/2}$$

$$= 4\pi a^2.$$

EXERCISE 1.7

1. Find the surface area of the solid generated by the revolution of astroid $x^{2/3} + y^{2/3}$ about x-axis.
2. Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about x-axis.
3. Determine the area of the surface generated by the revolution of the loops of the curve $r^2 = a^2 \cos 2\theta$ about initial line.
4. Find the volume of the solid formed by the revolution of the curve $y^2(a + x) = x^2(a - x)$ about x-axis.
5. The arc lying between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ of the curve $x = a \cos^3\theta$, $y = a \sin^3\theta$ rotates about the axis of x. Find the volume of the solid so generated.
6. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about x-axis.
7. Find the volume formed by the revolution of the curve $27ay^2 = 4(x - 3a)^2$ about x-axis.
8. Area bounded by x-axis, $y^2 = 4ax$ and the ordinate $x = 3a$ is revolved about x-axis. Find the volume generated.
9. Show that the volume of the solid generated by revolving the area included between the curves $y^2 = x^3$, $x^2 = y^3$ about x-axis is $\frac{5\pi}{28}$.
10. Find the surface and volume of the ellipsoid formed by the revolution of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis respectively.
11. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.
12. The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the area of the surface so generated.
13. For the curve $r^2 = a^2 \cos 2\theta$, prove that the volume of revolution about the initial line is $\frac{\pi a^2}{6\sqrt{2}}[3 \log(\sqrt{2} + 1) - \sqrt{2}]$

14. Find the volume of the solid generated by revolving the area included between the curve $\dfrac{y+8}{x} = x - 2$ and x-axis about the line $x + 5 = 0$.

15. Find the surface area of the solid formed by the revolution of the loop of the curve given by $3ay^2 = x(x - a)^2$.

16. Find the volume and surface of the solid generated by the revolution of the loop of the curve $x = t^2$, $y = \dfrac{t^3}{3}$ about x-axis.

17. A quadrant of a circle of radius 'a' revolves about its chord. Show that the volume of the spindle generated is $\dfrac{\pi a^3}{6\sqrt{2}}(10 - 3\pi)$.

Answers

1. $\dfrac{12}{5}\pi a^2$ **2.** $\dfrac{64}{3}\pi a^2$ **3.** $2\pi a^2(2 - \sqrt{2})$ **4.** $2\pi a^3\left[\log 2 - \dfrac{2}{3}\right]$

5. $\dfrac{16\pi a^2}{105}$ **6.** $\dfrac{\pi a^3}{12}$ **7.** $48\pi a^3$ **8.** $18\pi a^3$

10. $2\pi ab = \sqrt{1-e^2} + e^{-1}\sin^{-1} e$ and $\dfrac{4}{3}\pi ab^2$ **11.** $\dfrac{4}{5}\pi a^3$

12. $\pi a^2[3\sqrt{2} + \log(\sqrt{2} - 1)]$ **14.** 432π **15.** $\dfrac{4a}{\sqrt{3}}$

16. $\dfrac{3\pi}{4}, 3\pi$

INTERESTING FACTS

- Do you know, in electrical circuits, there exists a relationship between current and charge which can be calculated by this concept. (https://www.math24.net/integrals-electric-circuits)
- Engineering work in various industries utilizes the knowledge of this concept to find the Centre of Mass and Moment of Inertia of any object.
- Architects, while constructing any building use this concept.

VIDEO REFERENCES

Application of Definite Integral - I

APPLICATIONS TO REAL LIFE

- This concept is used in business and economics domain to calculate "*Lorenz curve and Gini coefficient*", and increase the total profit.
- Application in physics to find the mass and density of any object.
- It is also used to find average changes, volumes, error estimations and surface areas.

SUBJECTIVE SOLVED QUESTIONS
(HOTS)

Example 1. *Evaluate* $\int_{-3/2}^{10} \{2x\}\, dx$, *where* $\{.\}$ *denotes the fractional part of x.*

Solution. We know that $f(x) = \{2x\}$ is a periodic function with period $\frac{1}{2}$

Let $$I = \int_{-3/2}^{10} \{2x\}\, dx = \int_{-3(1/2)}^{20(1/2)} \{2x\}\, dx$$

$$= 23\int_0^{1/2} 2x\, dx \qquad (\text{as } \{2x\} = 2x - [2x] \text{ and when } x \in [0, 1/2], [2x] = 0)$$

$$= \left|23x^2\right|_0^{1/2}$$

$$= \frac{23}{4}$$

Remark. If $f(x)$ is a periodic function with period p, then $\int_{a/np}^{b/np} f(x)dx = \int_a^b f(x)dx, n \in I$.

Example 2. *Evaluate* $\int_{-1}^{1} x^3 \,.\, e^{x^4}\, dx.$

Solution. Let $f(x) = x^3 e^{x^4}$, then

$$f(-x) = (-x)^3 \,.\, e^{(-x)^4} = -x^3 e^{x^4} = -f(x)$$

Hence $f(x)$ is an odd function.

$$\therefore \qquad \int_{-1}^{1} f(x)dx = 0; \text{ or } \int_{-1}^{1} x^3 e^{x^4} dx = 0$$

Example 3. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Solution. Let $$I = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$$

Put $x^n = \sin^2\theta$ *i.e.* $x = \sin^{2/n}\theta$

$$dx = \frac{2}{n}\sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta d\theta$$

So, $$I = \frac{2}{n}\int_0^{\pi/2} \frac{\sin^{\left(\frac{2}{n}-1\right)}\theta\cos\theta\, d\theta}{\cos\theta}$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta\, d\theta$$

$$= \frac{2}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{n+2}{2n}\right)}$$

Example 4. *Evaluate* $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$.

Solution. Let
$$x = a\cos^2\theta + b\sin^2\theta$$
$$dx = 2a\cos\theta\sin\theta d\theta + 2b\sin\theta\cos\theta d\theta$$
$$= 2(b-a)\sin\theta\cos\theta d\theta$$
$$x - a = a\cos^2\theta + b\sin^2\theta - a$$
$$= (b-a)\sin^2\theta$$
$$b - x = b - a\cos^2\theta - b\sin^2\theta$$
$$= (b-a)\cos^2\theta$$

So, $$I = \int_0^{\pi/2} \frac{2(b-a)\sin\theta\cos\theta}{(b-a)\sin\theta\cos\theta}d\theta$$
$$= 2\int_0^{\pi/2} d\theta = \pi$$

Example 5. *Show, by means of a suitable substitution, that*

$$\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta = \frac{1}{2}\int_0^{\infty}\frac{t^{x-1}}{(1+t)^{x+y}}dt, x, y > 0$$

Solution. Let
$$\sin\theta = \frac{1}{\sqrt{1+z}}$$
$$\cos\theta d\theta = \frac{-1}{2}(1+z)^{-3/2}\, dz$$
$$\cos\theta = \frac{2^{1/2}}{\sqrt{1+z}}$$
$$I = \int_0^{\pi/2}\sin^{2x-1}\cos^{2y-1}\theta d\theta$$

$$= -\frac{1}{2}\int_{\infty}^{0} \frac{1}{(1+z)^{x-1/2}} \cdot \frac{z^{y-1}}{(1+z)^{y-1}} \cdot \frac{1}{(1+z)^{3/2}} dz$$

$$= \frac{1}{2}\int_{0}^{\infty} \frac{z^{y-1}}{(1+z)^{x+y}} dz = \frac{1}{2} B(y, x)$$

Since, $\quad B(x, y) = B(y, x)$

So, $\quad I = \frac{1}{2}\int_{0}^{\infty} \frac{z^{y-1}}{(1+z)^{x+y}} dz$

Example 6. *If* $I_n = \int_0^{\pi/2} \sin^n x\, dx$, *then show that* $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$.

Solution. Given $\quad I_n = \int_0^{\pi/2} \sin^n x\, dx$

$$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\sin^{n-2} x \,.\cos^2 x\, dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1)\int_0^{\pi/2} \sin^n x\, dx$$

$$I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

Example 7. *The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.*

Solution. The given parabola is $y^2 = 4ax$

Differentiating (1) w.r.t. x, we get $dy/dx = 2a/y$

$$\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1+\frac{4a^2}{y^2}\right\}}$$

$$= \sqrt{\left\{1+\frac{4a^2}{4ax}\right\}} = \sqrt{\left(\frac{x+a}{x}\right)}$$

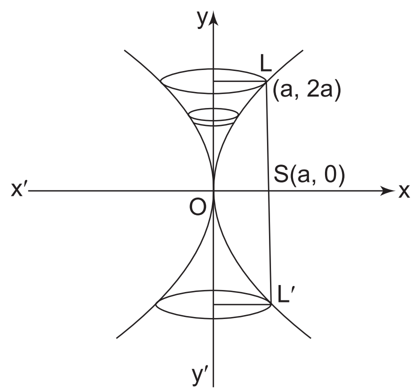


Fig. 1.32

The required curved surface is generated by the revolution of the arc LOL' (LSL' is the latus rectum), about the tangent at the vertex *i.e.*, y-axis. The curve is symmetrical about x-axis and for the arc OL, x varies from 0 to a.

$\therefore$ The required surface $\quad S = 2\int_0^a 2\pi x \frac{ds}{dx} dx$

$$= 4\pi\int_0^a x\sqrt{\left(\frac{x+a}{x}\right)}dx = 4\pi\int_0^a \sqrt{(x^2+ax)}dx$$

$$= 4\pi\int_0^a \sqrt{\left\{\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right\}}dx$$

$$= 4\pi\left[\frac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{(x^2+ax)} - \frac{1}{2}\cdot\frac{a^2}{4}\log\left\{\left(x+\frac{a}{2}\right)+\sqrt{(x^2+ax)}\right\}\right]_0^a$$

$$\left[\because \int\sqrt{(x^2-a^2)}dx = \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2\log\left\{x+\sqrt{(x^2-a^2)}\right\}\right]$$

$$= 4\pi\left[\frac{1}{2}\cdot\frac{3}{2}aa\sqrt{2} - \frac{1}{8}a^2\log\left\{\frac{3}{2}a + a\sqrt{2}\right\} + \frac{1}{2}a^2\log\left(\frac{1}{2}a\right)\right]$$

$$= 4\pi\left[\frac{3}{4}a^2\sqrt{2} - \frac{1}{8}a^2\log\left\{\left(\frac{3}{2}a + a\sqrt{2}\right)/\left(\frac{1}{2}a\right)\right\}\right]$$

$$= \pi a^2\left[3\sqrt{2} - \frac{1}{2}\log\left(3+2\sqrt{2}\right)\right]$$

$$= \pi a^2\left[3\sqrt{2} - \frac{1}{2}\log\left(\sqrt{2}+1\right)^2\right]$$

$$= \pi a^2\left[3\sqrt{2} - \log\left(\sqrt{2}+1\right)\right]$$

Example 8. *Find the area of the surface of the solid bounded by the cone $z = 3 - \sqrt{x^2+y^2}$ and the Paraboloid $z = 1 + x^2 + y^2$.*

Solution. Hint. Convert in polar coordinated then solve.

Example 9. *The part of the ellipse $x^2/a^2 + y^2/b^2 = 1$ cut off by a latus rectum revolves about the tangent at the nearer vertex. Find the volume of the reel thus generated.*

Solution. The given ellipse is $x^3/a^2 + y^2/b^2 = 1$. The focus of ellipse is $(ae, 0)$ where e is the eccentricity given by $e = \sqrt{1-\frac{b^2}{a^2}}$. The line segment passing through focus and intercepted by ellipse is called latus rectum.

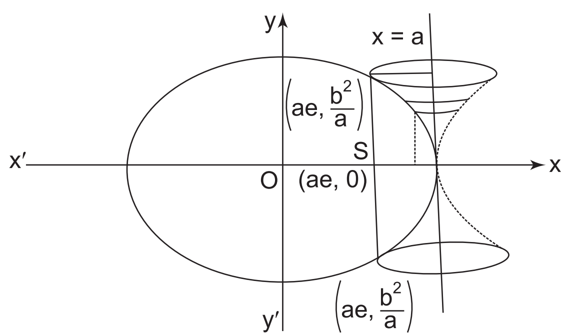


Fig. 1.33

Consider a volume element in the form of disc of radius $(a - x)$ and thickness dy. The volume of this element $\pi(a - x)^2\, dy$.

The volume of solid of revolution is given by

$$V = \int_{-b^2/a}^{b^2/a} \pi(a-x)^2\, dy$$

$$= 2\int_0^{b^2/a} \pi(a-x)^2\, dy \qquad \left[\text{where } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ i.e. } x^2 = \frac{a^2}{b^2}(b^2 - y^2)\right]$$

$$= 2\pi\int_0^{b^2/a} (a^2 - 2ax + x^2)dy$$

$$= 2\pi\int_0^{b^2/a} \left\{a^2 - 2b\,.\frac{a}{b}\sqrt{(b^2 - y^2)} + \frac{a^2}{b^2}(b^2 - y^2)\right\}dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^{b^2/a} \left\{2b^2 - 2b\sqrt{(b^2 - y^2)} - y^2\right\}dy$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2 y - 2b\left\{\frac{2}{2}y\sqrt{(b^2 - y^2)} + \frac{1}{2}b^2 \sin^{-1}\left(\frac{y}{b}\right)\right\} - \frac{y^3}{3}\right]_0^{b^2/a}$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2\,.\frac{b^2}{a} - 2b\left\{\frac{1}{2}\frac{b^2}{a}.\sqrt{\left(b^2 - \frac{b^4}{a^2}\right)} + \frac{1}{2}b^2 \sin^{-1}\frac{b}{a}\right\} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[2\frac{b^4}{a} - \frac{b^4}{a^2}.\sqrt{(a^2 - b^2)} - b^3 \sin^{-1}\frac{b}{a} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi b}{3a}\left\{6a^2 b - 3ab\sqrt{(a^2 - b^2)} - 3a^3 \sin^{-1}\frac{b}{a} - b^3\right\}$$

Example 10. *Find the volume of the solid generated by the revolution of the curve* $y = \dfrac{a^3}{(a^2 + x^2)}$ *about its asymptote.*

Solution. Answer: $\frac{1}{2}\pi^2 a^3$

Hint:

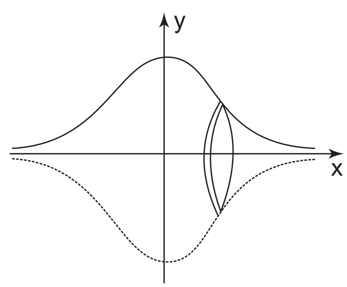


Fig. 1.34

Example 11. *Show that the volume of the solid generated by the revolution of the curve $(a-x)y^2 = a^2x$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.*

Solution. The given curve is $(a-x)y^2 = a^2x$. Its shape is as shown in the figure. Equating to zero, the coefficient of highest power of y, the asymptote parallel to the axis of y is $a - x = 0$ *i.e.*, $x = a$.

Consider a volume element in the form of disc of radius $(a-x)$ and thickness dy. The volume of this element $d\tau = \pi(a-x)^2\, dy$.

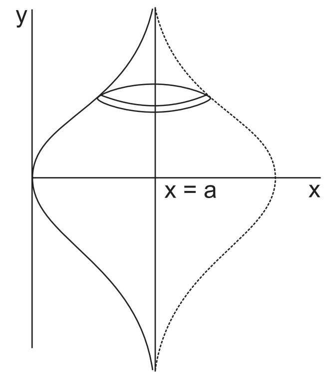


Fig. 1.35

$\therefore$ The required volume

$$V = 2\int_{y=0}^{\infty} \pi(a-x)^2\, dy$$

$$= 2\pi\int_0^{\infty}\left(a - \frac{ay^2}{y^2+a^2}\right)^2 dy \qquad \left[\because \text{ from (1), } x = \frac{ay^2}{y^2+a^2}\right]$$

$$= 2\pi a^6 \int_0^{\infty} \frac{dy}{(y^2+a^2)^2}$$

Now, put $y = a\tan\theta$ so that $dy = a\sec^2\theta d\theta$. When $y = 0$, $\theta = 0$ and when $y \to \infty$, $\theta \to \pi/2$.

Therefore, the required volume

$$= 2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\sec^4\theta} = 2\pi a^3 \int_0^{\pi/2} \cos^2 d\theta$$

$$= 2\pi a^3 . \frac{1}{2} . \frac{1}{2} \pi = \frac{1}{2}\pi^2 a^3$$

Example 12. *Discuss the convergence of the Beta function.* [*M.D.U. 2012; K.U.*]

Or

Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ *exists if and only if m, n are both positive.*

[*K.U. 2012, 08; M.D.U. 2008*]

Solution. Let $$I = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

The integral I is proper if $m \geq 1$ and $n \geq 1$ and so it is convergent of $m \geq 1$ and $n \geq 1$ Clearly 0 and 1 are the points of infinite discontinuity if $m < 1$ and $n < 1$ respectively. For $m < 1$ and $n < 1$, take a number $\frac{1}{2}$(say) between 0 and 1, so that we can write

$$I = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx + \int_{1/2}^1 x^{m-1}\,(1-x)^{n-1}\, dx$$

Let, $$I = I_1 + I_2 \qquad ...(1)$$

To discuss the convergence of $I_1 = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx$ *at* $x = 0$ *when* $m < 1$:

Here $$f(x) = x^{m-1}\,(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \qquad [\because \quad m < 1]$$

Take $g(x) = \dfrac{1}{x^{1-m}}$ so that $\lim\limits_{x\to 0^+} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0^+} (1-x)^{n-1} = 1$, which is finite and non-zero

$\therefore \quad I_1$ and $\int_0^{1/2} g(x)dx = \int_0^{1/2} \frac{1}{x^{1-m}} dx$ converge or diverge together.

But $\int_0^{1/2} \frac{1}{x^{1-m}} dx$ converges at $x = 0$ iff $1 - m < 1$ i.e., $m > 0$.

$\therefore \quad I_1$ converges at 0 iff $0 < m < 1$.

To discuss the convergence of $I_2 = \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ *at* $x = 1$ *when* $n < 1$:

Here $$f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}} \qquad [\because \quad n < 1]$$

Take $g(x) = \frac{1}{(1-x)^{1-n}}$ so that $\lim\limits_{x \to 1^-} \frac{f(x)}{g(x)} = \lim\limits_{x \to 1^-} x^{m-1} = 1$, which is finite and non-zero

$\therefore \quad I_2$ and $\int_{1/2}^{1} g(x)\,dx$ converge or diverge together.

But $\int_{1/2}^{1} g(x)\,dx = \int_{1/2}^{1} \frac{1}{(1-x)^{n-1}} dx$ converges at $x = 1$ iff $1 - n < 1$, i.e., $n > 0$

$\therefore \quad I_2$ converges at 1 iff $0 < n < 1$.

Therefore from (1), the integral I is convergent if and only if $0 < m < 1$ and $0 < n < 1$. Also it is a proper integral for $m \geq 1$ and $n \geq 1$.

Hence I is convergent iff m. n are both positive.

Example 13. *Discuss the convergence of Gamma function.* [*M.D.U. 2013, 11, 01*]

Or

Show that the integral $\int_0^{\infty} x^{n-1} e^{-x}\,dx$ *is convergent if* $n > 0$. [*M.D.U. 2012, 07*]

Solution. Let $$I = \int_0^{\infty} x^{n-1} e^{-x}\,dx$$
$$= \int_0^{1} x^{n-1} e^{-x}\,dx + \int_1^{\infty} x^{n-1} e^{-x}\,dx$$

Let $$I = I_1 + I_2 \qquad \ldots(1)$$

Here $$f(x) = x^{n-1} e^{-x} = \frac{e^{-x}}{x^{1-n}}$$

The integrand f has an infinite discontinuity at $x = 0$ in $[0, 1]$ if $n < 1$ and I_1 is a proper integral and hence convergent for $n \geq 1$.

To test the convergence of I_1 *at 0 when* $n < 1$:

Take $g(x) = \frac{1}{x^{1-n}}$ so that $\lim\limits_{x \to 0^+} \frac{f(x)}{g(x)} = \lim\limits_{x \to 0^+} e^{-x} = 1$, which is finite and non-zero.

But the integral $\int_0^1 g\,dx = \int_0^1 \frac{1}{x^{1-n}} dx$ is convergent if and only if $1 - n < 1$, i.e., $n > 0$.

Therefore by comparison test, the integral $\int_0^1 f\,dx$ converges for $0 < n < 1$. Also it is a proper integral for $n \geq 1$. Hence I_1 is convergent for all $n > 0$.

To test the convergence of I_2:

Take $g(x) = \dfrac{1}{x^2}$ so that $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{x^{n+1}}{e^x} = 0$ for all x.

Now $$\int_1^\infty g\,dx = \int_1^\infty \frac{1}{x^2}\,dx \text{ is convergent} \qquad [\because \ n = 2 > 1]$$

$\therefore$ By comparison test, the integral I_2 is also convergent for all n.

Hence by (1), we conclude that the given integral I is convergent if and only if $n > 0$.

SUMMARY

1. Radius of curvature $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$.

2. Coordinates of the centre of curvature $\bar{x} = x - \rho \sin\psi$, $\bar{y} = y + \rho \sin\psi$

$$\Rightarrow \quad \bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) \quad \text{and} \quad \bar{y} = y + \frac{1}{y_2}(1+y_1^2).$$

3. Evolute and Involute: The locus of the centre of curvature of a curve is called the evolute and the curve itself is called the involute.

4. If $f(x)$ is defined in the interval $[a, b]$, then the definite integral of $f(x)$ is written as

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

5. The integral $\int_a^b f(x)\,dx$ $f(x)$ is an improper integral, if either 'a' or 'b' or both 'a' and 'b' are infinite or the function $f(x)$ is unbounded on $[a, b]$.

6. Beta and Gamma functions

a. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges for $m, n > 0$

b. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$ converges for $n > 0$

c. $\Gamma(n + 1) = n\,\Gamma(n)$ and $\Gamma(n - 1) = n\,!$, if n is a positive integer

d. $\Gamma(1) = 1 = \Gamma(2), \Gamma(1/2) = \Gamma(\pi)$

e. $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

f. $$\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

g. $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta = \dfrac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \dfrac{1}{2}\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right)$

7. Surface area of solids of revolution

$$S = \int 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{(revolution about } x\text{-axis)}$$

$$S = \int 2\pi x\, ds = \int_a^b 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy \qquad \text{(revolution about } y\text{-axis)}$$

Volume of solids of revolution

Revolution about x-axis $= \int_a^b \pi y^2\, dx$

Revolution about y-axis $= \int_c^d \pi x^2\, dy$

OBJECTIVE QUESTIONS

1. Find the value of $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx$

a. $\dfrac{1}{77}$ b. $\dfrac{2}{77}$ c. $\dfrac{4}{77}$ d. $\dfrac{8}{77}$

2. The name of the evolute of an ellipse is

a. centroid b. astroid c. cycloid d. hyperboloid

3. Involute is also known as

a. evolute b. evolvent c. envelope d. tangent

4. What is the curvature of the curve $x^2 + y^2 = 25$?

a. 5 b. 25 c. 0.5 d. 0.2

5. What is the curvature of a straight line?

a. infinite b. one

c. zero d. length of the straight line

6. The value of the integral $\int_0^{\pi/2} [\tan^{-1}(\cot x) + \cot^{-1}(\tan x)]\, dx$ is

a. $\dfrac{\pi}{4}$ b. π c. $\dfrac{\pi^2}{4}$ d. $\dfrac{\pi^2}{2}$

7. If $I = \int_{-1}^{1} (x^7 + \cos^{-1} x)\, dx$, then $\cos I$ is equal to

a. 1 b. 0 c. –1 d. 1/2

8. The value of $\int_0^{\pi/2} \sin\theta \sqrt{\sin 2\theta}\, d\theta$ is ......

a. 1 b. 0 c. $\pi/2$ d. $\pi/4$

9. Which of the following is not a definition of Gamma function?

a. $\Gamma(n) = n\,!$ b. $\Gamma(n+1) = n\Gamma(n)$

c. $\Gamma(n) = \int_0^\infty x^{n-1}\, x^{-x}\, dx$ d. $\Gamma(n) = \int_0^1 \log\left(\frac{1}{y}\right)^{n-1}$

10. What is the value of $\int_0^1 \frac{x^2}{\sqrt{(1-x^4)}}\, dx$ is ....

a. $\frac{2\sqrt{\pi}\,\Gamma(5/4)}{\Gamma(1/4)}$ b. $\frac{2\pi\,\Gamma(3/4)}{\Gamma(1/4)}$ c. $\frac{2\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)}$ d. $\frac{2\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(5/4)}$

11. What is the value of $\Gamma(9/4)$?

a. $\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)$ b. $\frac{9}{4}\times\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)$ c. $\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{5}{4}\right)$ d. $\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)$

12. What is the value of $\Gamma(n)\,\Gamma(1-n)$?

a. $\frac{\pi}{\sin n\pi}$ b. $\frac{-\pi}{\sin n\pi}$ c. 0 d. $n\,!$

13. How much volume generated when the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is revolved about its minor axis?

a. $4ab$ cubic units b. $\frac{4}{3}a^2b$ cubic units c. $\frac{4}{3}ab$ cubic units d. 4 cubic units

14. How much volume generated when the region surrounded by $y = \sqrt{x}$, $y = 2$ and $y = 0$ is revolved about y-axis?

a. 32π cubic units b. $\frac{32\pi}{5}$ cubic units c. $\frac{32}{5}$ cubic units d. $\frac{5\pi}{32}$ cubic units

15. What is the area of the cardioid $y = a(1 + \cos\theta)$?

a. $\frac{3}{2}\pi a^2$ b. $3\pi a^2$ c. $\frac{3}{4}\pi a^2$ d. $\frac{3}{8}\pi a^2$

Answers

1. d	**2.** b	**3.** b	**4.** d
5. c	**6.** c	**7.** c	**8.** d
9. a	**10.** c	**11.** a	**12.** a
13. b	**14.** b	**15.** a	

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Define the osculating plane of curve at a point and from this definition, find its equation.

2. Show that the locus of the centre of curvature is an evolute, only when the curve is a plane.

3. Prove that the distance between corresponding points of two involutes is constant.

4. Evaluate $\int_{-2}^{3} |x^2 - 1|\, dx$.

5. Discuss the convergence of the integral $\int_0^1 x^{n-1} \log x\, dx$.

6. Prove that the area of the surface $z^2 = 2xy$ included between the planes $x = 0, x = a, y = 0, y = a$ is $4\sqrt{ab}\,\dfrac{a+b}{3\sqrt{2}}$.

7. Find the area of the surface $az = xy$ that lies inside the cylinder$(x^2 + y^2)^2 = 2a^2\, xy$.

8. Find the volume of the solid formed by revolving the cycloid about its base.

9. A quadrant of a circle of radius 'a' revolves about its chord. Find the volume of the spindle generated.

Answers

4. 28/3 **6.** convergent if $n > 0$ divergent if $n <= 0$

8. $1/9(20 - 3\pi)a^2$ **9.** $5\pi^2 a^3$ **10.** $\dfrac{\pi a^3 (10 - 3\pi)}{6\sqrt{2}}$

DID YOU KNOW?

Newton described his version of differential calculus as 'the method of fluxions'. He wrote a paper on fluxions in 1666, but like many of his works, it was not published until decades later. His magnum opus *Philosophiae naturalis principia mathematica* (Mathematical principles of natural philosophy) was published in 1687. This work includes his theories of motion and gravitation, but does not include much calculus explicitly — although there is some explanation of calculus at the beginning, and Newton certainly used calculus to formulate his theories. Nonetheless, Newton's 'method of fluxions' did not explicitly appear in print until 1693.

Leibniz, on the other hand, published his first paper on calculus in 1684 — and claimed to have discovered calculus in the 1670s. From the published record, at least, Leibniz seemed to have discovered calculus first.

While Newton and Leibniz initially had a cordial relationship, Leibniz and his followers did not take kindly to a statement made by the English mathematician John Wallis. With a rather xenophobic and quarrelsome character, Wallis fought priority disputes on behalf of English scientists throughout his life. In 1695, perhaps inadvertently, Wallis intimated that Leibniz learned about calculus from Newton — a claim now known to be false.

Then, offended by a statement of Leibniz that certain mathematical problems could only be solved by Leibniz's own version of the calculus, a mathematician named Fatio de Duiller in 1699 accused Leibniz of plagiarism. Things only went downhill from there. It did not help matters that Newton and Leibniz also disagreed on philosophical questions.

In 1712 the Royal Society in England wrote a report purporting to settle the matter — except, the whole investigation was effectively directed by Newton himself. The report found that Leibniz had concealed his knowledge of Newton's work — based on facts now known to be false. In response, Leibniz accused Newton and his followers of stealing Leibniz's own calculus and making errors in their applications of it. The dispute went on well after Leibniz's death in 1716, full of accusations and counter-accusations.

Nobody came out of the dispute well. Both Newton and Leibniz were capable of incredible mathematical discoveries, but their dispute demonstrated they were also capable of some rather less impressive behaviour.

PROJECT/PRACTICAL/ACTIVITIES

PROJECT

1. Create your script for computing the envelope of a rational family of lines to compute the equation of the evolute of the following curves:

i. the parabola $y = x^2$ parameterized as $x(t) = t$, $y(t) = t^2$.

ii. The ellipse $x^2 + 4y^2 = 4$ parameterized as $x(t)= \dfrac{8t}{1+4t^2}$, $y(t) = \dfrac{4t^2-1}{1+4t^2}$.

Plot these evolutes along with their corresponding curves .

2. Relate Beta function and String theory.

PRACTICAL

1. Plot 3-D image of Beta Function.

2. Sketch a graph and shade the area of the specified range for $\int_1^4 (x+6)\,dx$.

3. Use definite integral, find the shaded area for the given curves:

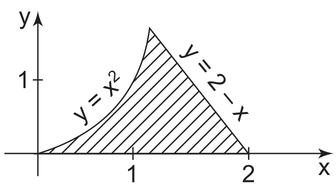
a. 

Fig. 1.36

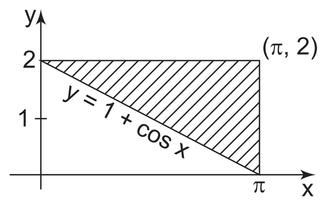
b. 

Fig. 1.37

4. Write a MATLAB Code to draw the graph of the evolute of a parabola.

ACTIVITY

i. How can the Gamma function interpolate the factorial function? (think and explain)

ii. How does the graph of Gamma function look like?
[**Hint:** Can use Python code.]

KNOW MORE

1. If 'f' is an even function and $\int_0^2 f(x)\,dx = k$, then $\int_{-1}^{1}\left(\dfrac{x^2-1}{x^2}\right) f\left(x+\dfrac{1}{x}\right)dx$ is equal to ......

a. 0 b. $2k$ c. k d. $4k$

2. The value of the integral $\int_{-5}^{5} (x - [x])\, dx$ is

a. 0 b. 5 c. 10 d. 15

3. $\int_0^2 x^{[x]}\, dx$ is equal to ......

a. $\frac{1}{2}$ b. $\frac{3}{2}$ c. $\frac{5}{2}$ d. $\frac{7}{2}$

4. Find the value of $\Gamma(0.1)\, \Gamma(0.2)\, \Gamma(0.3)$ ...... $\Gamma(0.9)$?

a. $\frac{(2\pi)^{9/2}}{\sqrt{10}}$ b. $\frac{(\pi)^2}{\sqrt{10}}$ c. $\frac{1}{2}$ d. $\frac{1}{\sqrt{2}}$

5. Examine the convergence of integral $A = \int_1^2 \frac{x}{\sqrt{x-1}}\, dx$, $B = \int_0^{\pi} \frac{dx}{1+\cos x}$.

6. Test the convergence of integral $\int_0^4 \frac{dx}{x(4-x)}$.

7. Let $\alpha = \int_0^{\infty} \frac{1}{1+t^2}\, dt$. Which of the following is true?

a. $\frac{d\alpha}{dt} = \frac{1}{1+t^2}$ b. α is a rational number

c. $\log(\alpha) = 1$ d. $\sin(\alpha) = 1$

Answers

1. b	**2.** b	**3.** c	**4.** a
5. A convergest to 8/3 and B diverges to $+\infty$		**6.** divergent	**7.** d

REFERENCES/SUGGESTED READINGS

1. Courant R. (1988). Differential and Integral Calculus , Wiley, NewYork.
2. Dass, H.K. Advanced Engineering Mathematics, S. Chand Publications.
3. Fleming, W.H. (1965). Functions of Several Variables, Addison Wesley Publishing Company, Reading, MA.
4. Garg Reena (2019). Engineering Mathematics-I, 2nd Edition, Khanna Book Publishing Co. (P)Ltd.
5. Grewal, B.S. Higher Engineering Mathematics, Khanna Publishers.
6. https://en.m.wikipedia.org/wiki/Evolute
7. Jain, R.K.; Iyengar, S.R.K. Advanced Engineering Mathematics, 2nd Edition, Narosa.
8. Knopp, K. (1947). Theory of Functions, Dover, NewYork.
9. Philip, Franklin (1940). A Treatise on Advanced Calculus, Wiley, Inc. NewYork.
10. Piskunov, N. (1969). Differential and Integral Calculus, MIR Publishers, Moscow.
11. Prasad, Chandrika (1967). Mathematics for Engineers, Pothishala Private Ltd.
12 Ram, Babu Engineering Mathematics, Pearson.
13. Thomas, G.B. and Finney, R.L. (2002). Calculus and Analytic Geometry, 9th edition, Pearson.
14. Tichmarsh, E.C. (1939). Theory of Functions, Oxford University Press, London.

2 Calculus II

UNIT SPECIFICS

This unit discusses the topics Rolle's theorem, it's geometrical interpretation, mean value theorems with their geometrical interpretation, Taylor's and Maclaurin's theorems with remainders, indeterminate forms and L'Hospital's rule (all types.), maxima and minima in length. The applications of various topics are discussed thoroughly and many solved examples are included for proper understanding of the topic. Many figures have been included so as to make students visualize the topics.

RATIONALE

Theorems lie at the core of mathematics. Theorems are often described as being "trivial", or "difficult", or "deep", or even "beautiful". These subjective judgments vary not only from person to person, but also with time and culture: for example, Rolle's theorem is used for analyzing the graphs of a company's yearly performance. Mean value theorem are often applied with motion problems such as throwing a ball into the air or else. These can be used as a mathematical tool in solving other problems related to computations.

Taylor Series are very useful to evaluate an approximation of many hard to calculate expressions. We use the L'Hospital Rule to solve the limits, it also has many applications in the real world, specially in statistics, physics, and engineering.

Everything in this world is based on the concept of maxima and minima, every time, everyone calculates the maximum and minimum value of every data.

PRE-REQUISITES

1. Concept of Continuity and differentiability.
2. Knowledge of special type of function such as mod, increasing function, decreasing function, open interval, closed interval.
3. Evaluation of limit, also applying it.
4. Aware with expansion of some functions like $\sin x$, $\cos x$, $\log(1 \pm x)$, e^x etc.

UNIT OUTCOMES

After completion of this unit, students will be able to:

U2-01: Apply the various Mean value theorems to prove the properties of a function comprised with its derivatives.

U2-02: Determine the asymptotic behaviour of function $f(x)$ as $x \to \infty$ and evaluate the limit using L'Hospital Rule.

U2-03: Analyse the behaviour of the function using Maxima-Minima.

U2-04: Learn about the expansion of series for the algebraic and transcendental function with Taylor' and Maclaurin's Theorem.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 2 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1- Weak Correlation; 2- Medium Correlation; 3- Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U2-01	1	2	–	–	–
U2-02	1	3	–	–	–
U2-03	–	3	–	–	1
U2-04	–	3	–	–	–

HISTORY

Earlier of the 17th century, a curve was generally described as a locus of points satisfying some geometric condition, and tangent lines were obtained through geometric construction. The relation between tangent lines to curves and the velocity of a moving particle was discovered in the late 1660s by Isaac Newton. Rolle's Theorem is a part, of the Mean Value Theorem. Bhaskara II (1114-1185), an Indian mathematician, is credited with being the first person to employ Rolle's Theorem and it was named after Michel Rolle (1652-1719), a French mathematician. The theorem was considered to be part of infinitesimal calculus and was not categorized under differential calculus until the 18th century. Lagrange provided a result only by using the first two conditions of Rolle's theorem. Hence it is called Lagrange's Mean-Value Theorem. Cauchy gave another mean value theorem in which he used two functions instead of one function as in the case of Rolle's theorem and Lagrange's Mean-Value Theorem, Lagrange's theorem is a particular of Cauchy Mean Value Theorem. Taylor's Theorem can be regarded as an extension of the Mean Value Theorem to "higher order" derivatives. L'Hospital's Rule was in fact discovered by Johann Bernoulli. In 1955, the L'Hospital–Bernoulli correspondence was published in Germany.

—Bhaksara II (1114-1185)

2.1 ROLLE'S THEOREM

Statement: Let f be a function defined on $[a, b]$ be such that

i. f is continuous on $[a, b]$ ii. f is differentiable on (a, b) iii. $f(a) = f(b)$

then there exist at least one real number c lying between a and b such that $f'(c) = 0$

Proof: Since f is continuous on $[a, b] \Rightarrow f$ is bounded on $[a, b]$ and attains its bounds.

Let $\qquad \underset{x\in[a,b]}{\text{Sup.}} f(x) = M \quad$ and $\quad \underset{x\in[a,b]}{\text{Inf.}} f(x) = m$

By the property of continuity, there exists $c, d \in [a, b]$ such that

$f(c) = M$ and $f(d) = m$ $\qquad$ [$\because$ if a function $f(x)$ is continuous on closed interval $[a, b]$, then it attains its supremum and infimum atleast once in $[a, b]$]

Two different cases arises:

Case I: When $M = m$ *i.e.* Sup. f = Inf. f

In this case, $\qquad f(x) = M(=m)$ for all $x \in [a, b]$

$\Rightarrow \quad f$ is a constant function over $[a, b]$

$\therefore \qquad f'(x) = 0$ for all $x \in [a, b]$

Hence $\qquad f'(c) = 0$ where $c \in (a, b)$

Case II: When $M \neq m$

Given that $\qquad f(a) = f(b)$

$\therefore \quad$ either M or m is different from $f(a) = f(b)$

Suppose that $\qquad M \neq f(a)$ and $M \neq f(b)$

As $\qquad f(c) = M$, then $c \neq a$, $c \neq b$ and therefore $a < c < b$

Since $\qquad f(c) = M = \underset{x\in[a,b]}{\text{Sup.}} f(x)$

Therefore, $\qquad f(x) \le f(c)$ for all $x \in [a, b]$ $\qquad$ ...(1)

From (1) $\qquad f(c - h) \le f(c)$

$\therefore \qquad f(c - h) - f(c) \le 0$

Dividing by $-h < 0$, we get

$$\frac{f(c-h) - f(c)}{-h} \ge 0$$

Taking limit as $h \to 0$

$$\lim_{h\to 0} \frac{f(c-h) - f(c)}{-h} \ge 0$$

$\Rightarrow \qquad Lf'(c) \ge 0$ $\qquad$ ...(2)

Again from (1), $\qquad f(c + h) \le f(c)$

$\therefore \qquad f(c + h) - f(c) \le 0$

Dividing by $h > 0$, we get

$$\frac{f(c+h) - f(c)}{h} \le 0$$

Taking limit as $h \to 0$

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \le 0$$

$\Rightarrow \qquad Rf'(c) \le 0$ $\qquad$ ...(3)

Since $f'(c)$ exists

$\therefore \qquad Lf'(c) = Rf'(c) = f'(c)$

This is possible only when $f'(c) = 0$.

Proceeding in the same manner, we can prove that $f'(d) = 0$, where $d \in (a, b)$.

Hence there exist atleast one $c \in (a, b)$ such that $f'(c) = 0$.

This completes the proof of the theorem.

Remark: In the proof of Rolle's theorem, we used some terms (like Sup. Inf.). Here we are giving the brief explanation of those.

1. **Least upper bound (Supremum): Definition:** Let S be a non-empty subset of $\Re$. A real number u is said to be a least upper bound or (l.u.b.) or supremum of S if
   i. $x \le u \ \forall \ x \in S$ *i.e.*, u is an upper bound of S.
   ii. If v is an upper bound of S, then $u \le v$.
2. **Greatest lower bound (Infimum): Definition:** Let S be a non-empty subset of $\Re$. A real number l is said to be a greatest lower bound or (g.l.b.) or infimum of S if
   i. $l \le x \ \forall \ x \in S$ *i.e.*, l is a lower bound of S.
   ii. If l' is any lower bound of S, then $l' \le l$. In other words any number greater than l is not a lower bound of S.

For example:

1. If $S = (0, 1)$, then clearly $0 < x < 1 \ \forall \ x \in S$
   Also, $x < 2 \ \forall \ x \in S$
   $\therefore$ 2, 3, 4... and so on are upper bounds of S but 1 is the least upper bound among them.
   $\therefore$ 1 is l.u.b. of S and $1 \notin S$.
   Similarly, $-1 < x \ \forall \ x \in S$
   $\therefore$ $-1, -2, -3, \ldots$ and so on are lower bounds of S but among all these lower bound, 0 is greatest.
   $\therefore$ 0 is g.l.b. of S and $0 \notin S$.
2. If $S = [0, 1]$, then $0 \le x \le 1 \ \forall \ x \in S$
   Here, 1 is l.u.b. of S and $1 \in S$.
   Also, 0 is g.l.b. of S and $0 \in S$.

Important Points:

i. l.u.b. or g.l.b. of a set if exist is unique.
ii. l.u.b. or g.l.b. of a set may or may not belong to that set.
iii. l.u.b. (supremum) of set may or may not exist like Sup. (N) does not exist. (here, N-set of Natural No.)
iv. g.l.b. (infimum) of a set may or may not exist like Inf (Z) does not exist (here Z-set of integers).

2.1.1 Geometrical Interpretation of Rolle's Theorem

Let the curve $y = f(x)$

i. continuous on $[a, b]$
ii. derivable on (a, b)
iii. $f(a) = f(b)$

This imply that there exists at least one point $c \in (a, b)$ at which tangent is parallel to x-axis.

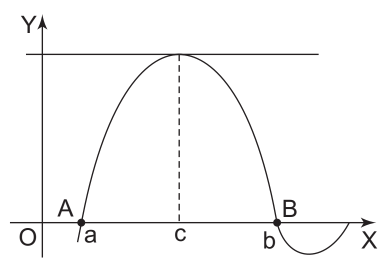


Fig. 2.1

For example:

1. $f(x) = [x]$, greatest integer functions on $[0, 3]$.
f is not continuous at $x = 1, 2, 3$ (break in graph).
$\therefore$ Rolle's theorem does not hold good.

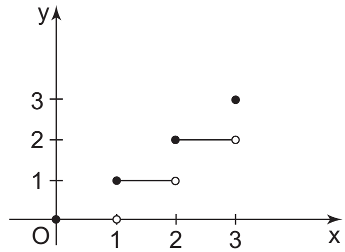


Fig. 2.2

2. $f(x) = x$ in $[-1, 1]$
f is continuous on $[-1, 1]$, f is derivable on $(-1, 1)$ but $f(-1) \neq f(1)$
$\therefore$ Rolle's theorem does not hold good.

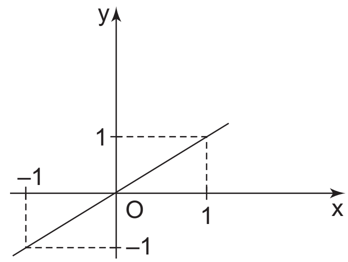


Fig. 2.3

3. $f(x) = |x|$ in $[-1, 1]$
f is continuous on $[-1, 1]$
but f is not derivable at $x = 0$
$\therefore$ Rolle's theorem does not hold good.

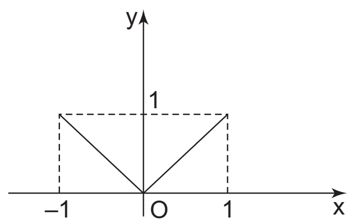


Fig. 2.4

SOME SOLVED EXAMPLES

Example 2.1. *Verify Rolle's theorem for $f(x) = x^3 - 9x^2 + 26x - 24$ in $[2, 4]$.*

Solution. Here, $\quad f(x) = x^3 - 9x^2 + 26x - 24.$

Given $f(x)$ is a polynomial of x and therefore continuous and derivable for all x.

$\Rightarrow$ a. $f(x)$ is continuous on $[2, 4]$

b. $f(x)$ is derivable on $(2, 4)$

c. $f(2) = (2)^3 - 9(2)^2 + 26(2) - 24 = 0$

$f(4) = (4)^3 - 9(4)^2 + 26(4) - 24 = 0$

All three conditions of Rolle's theorem are satisfied.

Hence there must exists atleast one $c \in (2, 4)$ such that $f'(c) = 0$

We have, $\quad f'(x) = 3x^2 - 18x + 26$

$\Rightarrow \quad f'(c) = 3c^2 - 18c + 26$

Thus, $\quad f'(c) = 0$

$$\Rightarrow \qquad c = \frac{18 \pm \sqrt{324 - 312}}{6}$$

$$= 3 \pm \frac{1}{\sqrt{3}}$$

$$c = 3 \pm \frac{1}{\sqrt{3}} \in (2, 4)$$

Hence, Rolle's theorem is verified.

Example 2.2. *Verify Rolle's theorem for f(x) = cos 2x in* $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Solution. Here $f(x) = \cos 2x$

a. As we know that cosine function is continuous for all values of x and hence $f(x)$ is continuous in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

b. $f'(x) = -2 \sin 2x$ = finite, defined

$\therefore$ $f(x)$ is derivable on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

c. Now, $$f'\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{2} = 0$$

and $$f'\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right)$$

$$= \cos \frac{\pi}{2} = 0$$

$$\therefore \qquad f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$$

All the conditions of Rolle's theorem are satisfied.

Hence, there must exists atleast one value of $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $f'(c) = 0$

Now $$f'(c) = -2 \sin 2c = 0 \quad \Rightarrow \quad \sin 2c = 0$$

$$\sin 2c = \sin 0$$

$$\Rightarrow \qquad 2c = 0 \quad \Rightarrow \quad c = 0$$

$$c = 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Hence, Rolle's theorem is verified.

Example 2.3. *Verify Rolle's theorem for* $f(x) = x(x + 3)e^{-x/2}$ *in [–3, 0].*

Solution. Here $\qquad f(x) = x(x + 3)e^{-x/2}$

a. We know that a function $x(x + 3)$ being a polynomial function and $e^{-x/2}$, the exponential function, both are continuous everywhere.

Thus there product is also continuous in [–3, 0].

b. $$f'(x) = xe^{-x/2} + (x+3)e^{-x/2} - \frac{x(x+3)}{2}\, e^{-x/2}$$
$$= \left(2x + 3 - \frac{x(x+3)}{2}\right)e^{-x/2}$$
$$= \left(\frac{x+6-x^2}{2}\right)e^{-x/2}$$

$f'(x)$ exists uniquely on $(-3, 0)$ and $f(x)$ is derivable on $(-3, 0)$

c. $$f(-3) = -3\,(-3+3)e^{3/2} = 0$$

and $$f(0) = 0\,(0+3)e^{0/2} = 0$$

Thus $$f(-3) = f(0)$$

Thus, all the conditions of Rolle's theorem are satisfied.

Hence, there must exist atleast one point c in $(-3, 0)$ such that $f'(c) = 0$,

$$f'(c) = \left(\frac{c+6-c^2}{2}\right)e^{-c/2} = 0$$

$\Rightarrow$ $$\frac{c+6-c^2}{2} = 0 \qquad [\because \; e^{-c/2} \neq 0$$

$\Rightarrow$ $$c^2 - c - 6 = 0$$

$\Rightarrow$ $$c = 3, -2$$

Now $$c = -2 \in (-3, 0)$$

Hence Rolle's theorem is verified.

Example 2.4. *Examine the applicability of Rolle's theorem for the function* $f(x) = \begin{cases} -4x+5 & 0 \le x \le 1 \\ 2x-3 & 1 < x \le 2 \end{cases}$.

Solution. Here, $$f(x) = \begin{cases} -4x+5 & 0 \le x \le 1 \\ 2x-3 & 1 < x \le 2 \end{cases}$$

We have, $$f(1) = 1$$

Continuity at, $$x = 1$$

$$Rf(1) = \lim_{x \to 1^+} (2x-3) = \lim_{h \to 0} 2(1+h) - 3$$
$$= 2 - 3 = -1$$
$$Lf(1) = \lim_{x \to 1^-} (-4x+5) = \lim_{h \to 0} -4\,(1-h) + 5$$
$$= -4 + 5 = 1$$

Thus, $$Rf(1) \neq Lf(1)$$

$\therefore$ $f(x)$ is not continuous at $x = 1 \in [0, 2]$

Hence, Rolle's theorem is not applicable.

Example 2.5. *Verify Rolle's theorem for* $f(x) = \sqrt{4-x^2}$ *in* $[-2, 2]$.

Solution. Here $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$. $f(x)$ is a square root of a polynomial of x and therefore continuous for all x.

i. $f(x)$ is continuous on $[-2, 2]$

ii. $f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$ defined everywhere except where $4 - x^2 = 0$ *i.e.*, $x = \pm 2$.

Thus, $f'(x)$ is derivable in $R - \{-2, 2\}$

$\therefore$ $f'(x)$ is derivable on $(-2, 2)$.

iii. Now, $$f(-2) = \sqrt{4-(-2)^2} = \sqrt{4-4} = 0$$

$$f(2) = \sqrt{4-(2)^2} = \sqrt{4-4} = 0$$

Thus, $$f(-2) = f(2)$$

All the conditions of Rolle's theorem are satisfied.

Hence, there must exist atleast one value of $c \in (-2, 2)$ such that $f'(c) = 0$

$$\Rightarrow \qquad f'(c) = \frac{-c}{\sqrt{4-c^2}} = 0$$

$$\Rightarrow \qquad c = 0 \in (-2, 2)$$

Hence, Rolle's theorem is verified.

Example 2.6. *Find a point $c \in (-1, 1)$ using Rolle's theorem for the function*

$$f(x) = \log (x^2 + 2) - \log 3 \text{ in } [-1, 1].$$

Solution. Here $f(x) = \log (x^2 + 2) - \log 3$.

a. As we know that logarithmic functions are continuous for all x and log 3 is a constant, so $f(x)$ is continuous for all x, therefore, continuous in $[-1, 1]$.

b. $$f'(x) = \frac{2x}{x^2+2} \quad (\text{exists for all } x \in \mathbb{R})$$

$\therefore$ $f'(x)$ is derivable on $(-1, 1)$

c. Now, $$f(-1) = \log [(-1)^2 + 2] - \log 3$$
$$= \log 3 - \log 3 = 0$$

and $$f(1) = \log [(1)^2 + 2] - \log 3$$
$$= \log 3 - \log 3 = 0$$

$$\Rightarrow \qquad f(-1) = f(1)$$

Thus, all conditions of Rolle's theorem are satisfied.

Hence there must exist atleast one $c \in (-1, 1)$ such that $f'(c) = 0$

$$f'(c) = \frac{2c}{c^2+2} = 0$$

$$\Rightarrow \qquad c = 0 \in (-1, 1).$$

EXERCISE 2.1

1. Verify Rolle's theorem for the following functions in the given intervals:

a. $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$

b. $f(x) = x^3 + 3x^2 - 24x - 80$ in $[-4, 5]$

c. $f(x) = \dfrac{x^2 - 3x - 4}{x - 5}$ in $[-1, 4]$

d. $f(x) = \cos 2\left(x - \dfrac{\pi}{4}\right)$ in $\left[0, -\dfrac{\pi}{2}\right]$

e. $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

f. $f(x) = e^{1-x^2}$ in $[-1, 1]$

g. $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$

h. $f(x) = e^x \cos x$ in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

i. $f(x) = \tan x$ in $[0, \pi]$

j. $f(x) = (x^2 - 4x + 3)e^{2x}$ in $[1, 3]$

2. Examine the applicability of Rolle's theorem for the following functions:

a. $f(x) = (x-1)^{2/5}$ in $[0, 3]$

b. $f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$

c. $f(x) = |x|$ in $[-1, 1]$

Answers

1. a. $c = 2 \pm \dfrac{1}{\sqrt{3}}$ b. $c = 2$ c. $c = 5 - \sqrt{6}$ d. $c = \dfrac{\pi}{4}$

e. $c = \cos^{-1}\dfrac{1 \pm \sqrt{33}}{8}$ f. $c = 0$ g. $c = \dfrac{\pi}{4}$ h. $c = \dfrac{\pi}{4}$

i. Not applicable j. $c = \dfrac{3+\sqrt{5}}{2}$

2. a. Not applicable b. Not applicable c. Not applicable

2.1.2 Lagrange's Mean Value Theorem

Statement: If a function $f : [a, b] \to \mathbb{R}$ be such that

i. $f(x)$ is continuous on closed interval $[a, b]$

ii. $f(x)$ is derivable on open interval (a, b),

then there exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof: Let us define a function $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) + Ax$, $x \in (a, b)$ where A is a constant to be determined such that

$$\phi(a) = \phi(b) \qquad ...(1)$$

Now,

$$\phi(a) = f(a) + Aa$$

$$\phi(b) = f(b) + Ab$$

Using (1), we have

$$f(a) + Aa = f(b) + Ab$$

$$A(a - b) = f(b) - f(a)$$

$$A = \frac{f(b) - f(a)}{a - b} \qquad ...(2)$$

Now,

i. ϕ is continuous on $[a, b]$, since f is continuous on $[a, b]$ and Ax is polynomial in x is continuous on $[a, b]$.

ii. ϕ is derivable on (a, b), since f is derivable at each point of (a, b) and also Ax.

iii. $\phi(a) = \phi(b)$

$\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exists atleast one $c \in (a, b)$ such that $\phi'(c) = 0$

$$\phi(x) = f(x) + Ax$$

$$\Rightarrow \quad \phi'(x) = f'(x) + A$$

$$\Rightarrow \quad \phi'(c) = f'(c) + A$$

Now, $\phi'(c) = 0$

$$\Rightarrow \quad f'(c) + A = 0 \quad \Rightarrow \quad f'(c) = -A$$

Using (2), we have, $f'(c) = \dfrac{f(b) - f(a)}{b - a}, c \in (a, b)$

This completes the proof of the theorem.

2.1.3 Geometrical Interpretation of Lagrange's Mean Value Theorem

Let a function f has a graph which is

i. continuous on $[a, b]$

ii. differentiable on (a, b)

As curve AB has a tangent at every point, then there exist a point on the curve other than A and B where tangent is parallel to line segment joining the points $(a, f(a))$ and $(b, f(b))$.

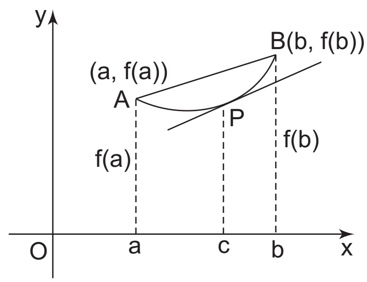


Fig. 2.5

Remarks: Rolle's theorem is a special case of Lagrange's mean value theorem.

In addition with two conditions of mean value theorem, if

$$f(a) = f(b)$$

then $f(b) - f(a) = 0$

and hence $f'(c) = 0$

In Geometrical Interpretation, there is a point on curve at which tangent is parallel to x-axis.

SOME SOLVED EXAMPLES

Example 2.7. *Verify Lagrange's mean value theorem for* $f(x) = x + \dfrac{1}{x}$ *in [1, 3].*

Solution. Here, $f(x) = x + \dfrac{1}{x}$

i. $f(x)$ is polynomial in x and continuous for all values of $x \in \mathbb{R} - \{0\}$

$\therefore$ $f(x)$ is continuous on $[1, 3]$

ii. $f'(x) = 1 - \dfrac{1}{x^2}$ exists for all $x \in (1, 3)$

$\therefore$ $f(x)$ is derivable in $(1, 3)$

Both conditions of Lagrange's mean value theorem are satisfied.

Hence, there must exist atleast one $c \in (1, 3)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad \text{...(1) [Here, } b = 3, a = 1]$$

Putting $f(b), f(a), f'(c)$ in (1), we have

$$\frac{\frac{10}{3}-2}{3-1} = 1 - \frac{1}{c^2}$$

$$\Rightarrow \qquad \frac{1}{c^2} = \frac{1}{3} \quad \Rightarrow \quad c = \pm\sqrt{3}$$

$$c = \sqrt{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

Example 2.8. *Show that* $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$, $x > 0$.

Solution. Consider $\quad f(x) = \log(1+x) - \left(x - \dfrac{x^2}{2}\right)$

Differentiating w.r.t. x,

$$f'(x) = \frac{1}{1+x} - (1-x)$$

$$= \frac{x^2}{1+x} > 0 \qquad [\because \quad x > 0]$$

Hence $f(x)$ is increasing function for all $x > 0$

Also $f(0) = 0$

Hence, $f(x) > 0$ for $x > 0$

Thus, $\qquad \log(1+x) > x - \dfrac{x^2}{2} \qquad$ ...(1)

Let $\qquad g(x) = x - \dfrac{x^2}{2(1+x)} - \log(1+x)$

Differentiating w.r.t x,

$$g'(x) = 1 - \frac{2x+x^2}{2(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{x^2}{2(1+x)^2} = \frac{1}{2}\left(\frac{x}{1+x}\right)^2 > 0$$

$\therefore \quad g(x)$ is an increasing function for all $x > 0$

Also $g(0) = 0$

Hence, $g(x) > 0$ for $x > 0$

Thus, $\qquad x - \dfrac{x^2}{2(1+x)} > \log(1+x) \qquad$ ...(2)

From (1) and (2), we have

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, x > 0.$$

Example 2.9. *Examine the validity of Lagrange's mean value theorem for the function*

$$f(x) = \begin{cases} 1+3x, & x \le 1 \\ 2x^2+2, & x > 1 \end{cases} \text{ in } [0, 3].$$

Solution. Here $f(x) \begin{cases} 1+3x, & x \le 1 \\ 2x^2+2, & x > 1 \end{cases}$ in $[0, 3]$

i. $f(x)$ is continuous on $[0, 3] - \{1\}$ being a polynomial function.

Continuity at $x = 1$

$$R f(1) = \lim_{x \to 1^+} 2x^2 + 2 = \lim_{h \to 0} 2(1+h)^2 + 2 = 4$$

$$L f(1) = \lim_{x \to 1^-} 1 + 3x = \lim_{h \to 0} 1 + 3(1-h) = 4$$

Also, $f(1) = 4$

We have $R f(1) = L f(1) = f(1)$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$

$\therefore$ $f(x)$ is continuous on $[0, 3]$

ii. $$f'(x) = \begin{cases} 3, & x \le 1 \\ 4x, & x > 1 \end{cases} \text{ in } [0, 3]$$

Differentiability at $x = 1$

$$R f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{4(1+h) - 4}{h} = 4$$

$$L f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3-4}{h} = \text{does not exist}$$

$f'(x)$ does not exist for $x = 1 \in (0, 3)$

$\Rightarrow$ $f(x)$ is not derivable on $(0, 3)$

Hence Lagrange's mean value theorem is not applicable.

Example 2.10. *Verify Lagrange's mean value theorem for $f(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$.*

Solution. Here $f(x) = \cos x$

i. As we know, cosine function is continuous for all value of x.

$\therefore$ $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$

ii. $f'(x) = -\sin x$ (finite and definite)

$\therefore$ $f(x)$ is derivable in $(0, \pi/2)$

Both conditions of Lagrange's mean value theorem are satisfied.

Hence there must exist atleast one $c \in \left(0, \frac{\pi}{2}\right)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c), \text{ here } b = \pi/2,\ a = 0$$

$$f(\pi/2) = \cos \pi/2 = 0,\ f(0) = \cos 0 = 1$$

Putting values,

$$\Rightarrow \quad \frac{0-1}{\frac{\pi}{2}-0} = -\sin c$$

$$\Rightarrow \quad c = \sin^{-1}(2/\pi)$$

$$\Rightarrow \quad c = \sin^{-1}(0.636) \in \left(0, \frac{\pi}{2}\right)$$

Hence Lagrange's mean value theorem is satisfied.

Example 2.11. *Show that* $\frac{x}{1+x^2} < \tan^{-1} x < x,\ x > 0.$

Solution. Consider $\quad f(x) = \tan^{-1}x - \frac{x}{1+x^2}$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{2x^2}{(1+x^2)^2} > 0 \ \forall\ x > 0$$

Hence $f(x)$ is a increasing function for all $x > 0$

Also $\quad f(0) = 0 \qquad$ (as $\tan^{-1} 0 - 0 = 0$)

Hence $\quad f(x) > 0 \ \forall\ x > 0$

Thus, $\quad \tan^{-1}x > \frac{x}{1+x^2} \qquad$ ...(1)

Let $\quad g(x) = x - \tan^{-1}x$

Differentiating w.r.t. x,

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \ \forall\ x > 0$$

$\therefore$ $g(x)$ is an increasing function for all $x > 0$

Also, $\quad g(0) = 0 - \tan^{-1} 0 = 0$

Hence $\quad g(x) > 0 \ \forall\ x > 0$

Thus, $\quad x > \tan^{-1} x \ \forall\ x > 0 \qquad$ ...(2)

Thus, from (1) and (2), we have

$$\frac{x}{1+x^2} < \tan^{-1} x < x,\ x > 0.$$

EXERCISE 2.2

1. Verify Lagrange's Mean Value theorem for the following functions in given intervals.

a. $f(x) = 2x^2 - 3x + 1$ in $[1, 3]$

b. $f(x) = x(x-1)(x-2)$ in $\left[0, \frac{1}{2}\right]$

c. $f(x) = \dfrac{1}{4x-1}$ in $[1, 4]$

d. $f(x) = \sqrt{25 - x^2}$ in $[-3, 4]$

e. $f(x) = \log x$ in $[1, e]$

f. $f(x) = x - 2\sin x$ in $[-\pi, \pi]$

2. Examine the applicability of Lagrange's mean value theorem for following functions:

a. $f(x) = |x|$ in $[-1, 1]$

b. $f(x) = \beta$ (constant function) in $[a, b]$

c. $f(x) = x^{1/3}$ in $[-1, 1]$

d. $f(x) = |x + 2|$ in $[-3, 4]$

3. Using Lagrange's Mean Value theorem, prove that

a. $\dfrac{x^2}{2} < x - \log(1+x) < \dfrac{x^2}{2(1+x)}$ on $[-1, 0]$

b. $\dfrac{x}{1+x} < \log(1+x) < x, x > 0$

c. $1 + x < e^x < 1 + xe^x$ for all $x > 0$

4. Show that $\dfrac{y-x}{1+y^2} < \tan^{-1} y - \tan^{-1} x < \dfrac{y-x}{1+x^2}$ if $0 < x < y$ and deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

Answers

1. a. $c = 2$ b. $c = \dfrac{6 - \sqrt{21}}{6}$ c. $c = \dfrac{1 + 3\sqrt{5}}{4}$ d. $c = \pm\dfrac{1}{\sqrt{2}}$

e. $c = e - 1$ f. $c = \pm\pi/3$

2. a. Not applicable b. Applicable c. Not applicable d. Not applicable

2.1.4 Cauchy Mean Value Theorem

Statement: Let the function $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that

i. f and g are both continuous on $[a, b]$

ii. f and g are both differentiable on (a, b)

iii. $g'(x) \neq 0$ for all $x \in (a, b)$, then there exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Suppose $g(a) = g(b)$, then g would satisfy all the conditions of Rolle's theorem.

So, there exist atleast one point $c \in (a, b)$ such that $g'(c) = 0$

But this contradicts the given fact that $g'(x) \neq 0\ \forall\ x \in (a, b)$, so our supposition is wrong and $g(a) \neq g(b)$.

Now define a function $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) = f(x) + Ag(x)$, $x \in (a, b)$ where A is constant and to be determined in such a way that,

$$\phi(a) = \phi(b) \quad \text{...(1)}$$

Now, $\phi(a) = f(a) + Ag(a)$

$$\phi(b) = f(b) + Ag(b)$$

Using (1), we have

$$f(a) + Ag(a) = f(b) + Ag(b)$$

$$\Rightarrow \quad A[g(a) - g(b)] = f(b) - f(a)$$

$$\Rightarrow \quad A = \frac{f(b) - f(a)}{g(a) - g(b)} \quad \text{...(2)}$$

Now,

i. ϕ is continuous on $[a, b]$, since f and g both are continuous on $[a, b]$ and A being a constant is also continuous on $[a, b]$

ii. ϕ is derivable on (a, b), since f and g both are differentiable on (a, b) and A being a constant is also derivable on (a, b)

iii. Also, $\phi(a) = \phi(b)$

Thus, $\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exist atleast one $c \in (a, b)$ such that $\phi'(c) = 0$

$$\phi(x) = f(x) + Ag(x)$$

$$\Rightarrow \quad \phi'(x) = f'(x) + Ag'(x)$$

$$\Rightarrow \quad \phi'(c) = f'(c) + Ag'(c)$$

Now, $\phi'(c) = 0$

$$\Rightarrow \quad f'(c) + Ag'(c) = 0$$

$$\Rightarrow \quad f'(c) = -Ag'(c)$$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)} = -A$$

Using (2), we have, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$, $c \in (a, b)$

Hence, theorem is proved.

Remark: Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem by taking $g(x) = x$, $x \in [a, b]$.

2.1.5 Geometrical Interpretation of Cauchy's Mean Value Theorem

Let $x = f(t)$ and $y = g(t)$ be parametric curve, $t \in (a, b)$

i. f, g continuous on $[a, b]$

ii. f, g derivable on (a, b)

iii. $g'(x) \neq 0$ on (a, b)

then there exist atleast one $c \in (a, b)$ at which tangent is parallel to AB.

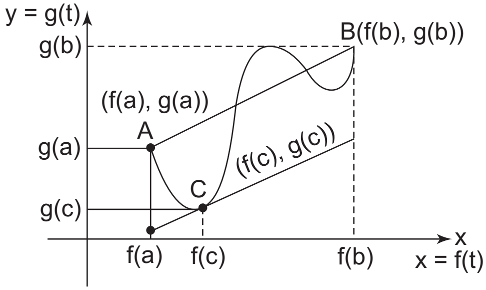


Fig. 2.6

SOME SOLVED EXAMPLES

Example 2.12. *Verify Cauchy's mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ on [0, 1].*

Solution. Here, $f(x) = e^x, g(x) = e^{-x}$

i. f and g are continuous function on [0, 1]

ii. $f'(x) = e^x, g'(x) = -e^{-x}$ are differentiable on (0, 1)

iii. $g'(x) = -e^{-x} \neq 0 \ \forall \ x \in (0, 1)$,

then there exist atleast one $c \in (0, 1)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad [\text{here } b = 1, a = 0] \qquad \text{...(1)}$$

So $\qquad f(1) = e^1 = e, f(0) = e^0 = 1$

and $\qquad g(1) = e^{-1} = \dfrac{1}{e}, g(0) = e^{-0} = 1$

Putting all values in (1), we have

$$\frac{e-1}{\frac{1}{e}-1} = \frac{e^c}{-e^{-c}}$$

$\Rightarrow \qquad 1 = e^{2c-1}$

$\Rightarrow \qquad e^0 = e^{2c-1}$

$\Rightarrow \qquad 0 = 2c - 1 \quad \Rightarrow \quad c = \dfrac{1}{2} \in (0, 1)$

Hence Cauchy's mean value theorem is verified.

Example 2.13. *Let the function f be continuous in $[a, b]$ and derivable in (a, b). Show that there exists a number c in (a, b) such that $2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2]$.*

Solution. i. f is continuous in $[a, b]$

ii. f' is derivable in (a, b)

Both conditions of Lagrange's mean value theorem are satisfied.

Hence there exist atleast one $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \qquad \frac{f(a)-f(b)}{a-b} = f'(c)$$

$$\Rightarrow \qquad f(a) - f(b) = f'(c)\,(a - b) \qquad ...(1)$$

Now, given $2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2]$

Using (1), we have

$$2c\,[(a - b)\,f'(c)] = f'(c)\,[(a - b)\,(a + b)]$$

$$\Rightarrow \qquad 2c = a + b$$

$$\Rightarrow \qquad c = \frac{a+b}{2} \in (a, b)$$

Hence, there exist a number $c \in (a, b)$.

Example 2.14. *Find 'c' in the Cauchy mean value theorem for the function*

$$f(x) = \frac{1}{x},\ g(x) = x^2 - 4 \text{ in } [1, 2] \text{ using } \sqrt[3]{3} = 1.44.$$

Solution. Here, $f(x) = \dfrac{1}{x}$, $g(x) = x^2 - 4$ in $[1, 2]$

i. f is continuous function for all $x \in R - \{0\}$ $\qquad$ [$\because$ f is not defined at $x = 0$]

and g being a polynomial function is continuous everywhere.

$\therefore$ $f(x)$ and $g(x)$ are continuous function on $[1, 2]$

ii. $f'(x) = -\dfrac{1}{x^2}$, $g'(x) = 2x$ are derivable on $(1, 2)$

iii. $g'(x) = 2x \neq 0\ \forall\ x \in (1, 2)$,

then there exist atleast one $c \in (1, 2)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad ...(1) \quad \text{here } a = 1, b = 2$$

So $\qquad f(2) = 1/2, f(1) = 1, g(2) = 0, g(1) = -3$

Putting all values in (1), we have

$$\frac{\frac{1}{2}-1}{0-(-3)} = \frac{-1/c^2}{2c}$$

$$\Rightarrow \qquad c^3 = 3$$

$$\Rightarrow \qquad c = \sqrt[3]{3} = 1.44 \in (1, 2).$$

Example 2.15. *Verify Cauchy mean value theorem for the function $f(x) = x^2$, $g(x) = x^4$ in $[a, b]$, where $a > 0, b > 0$.*

Solution. Here $\qquad f(x) = x^2, g(x) = x^4$

i. since f and g are polynomial functions of x, therefore continuous everywhere.

$\therefore$ $f(x)$ and $g(x)$ are continuous on $[a, b]$

ii. $\qquad f'(x) = 2x, g'(x) = 4x^3$

$f'(x)$ and $g'(x)$ are again polynomial function and hence derivable everywhere.

$\therefore$ $f'(x)$ and $g'(x)$ are derivable on (a, b)

iii. $\qquad g'(x) = 4x^3 \neq 0\ \forall\ x \in (a, b)$, $[a > 0, b > 0]$

Thus f and g satisfies all the conditions of Cauchy mean value theorem.

$\therefore$ there must exist atleast one $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \qquad \text{[here } a = a, b = b\text{]}$$

So, $f(b) = b^2, f(a) = a^2, g(a) = a^4, g(b) = b^4$

$$\Rightarrow \quad \frac{b^2-a^2}{b^4-a^4} = \frac{2c}{4c^3}$$

$$\Rightarrow \quad \frac{1}{b^2+a^2} = \frac{1}{2c^2}$$

$$\Rightarrow \quad c^2 = \frac{b^2+a^2}{2}$$

$$\Rightarrow \quad c = \pm\sqrt{\frac{a^2+b^2}{2}} \in (a, b)$$

Hence, Cauchy mean value theorem is verified.

EXERCISE 2.3

1. Verify Cauchy's Mean Value Theorem for the following functions:

a. $f(x) = \sin x, g(x) = \cos x$ in $\left[-\frac{\pi}{2}, 0\right]$

b. $f(x) = x^2, g(x) = x^3$ in $[1, 2]$

c. $f(x) = \sqrt{x}, g(x) = \frac{1}{\sqrt{x}}$ in $[1, 3]$

d. $f(x) = \log x, g(x) = \frac{1}{x}$ in $[1, e]$

e. $f(x) = (1 + x)^{3/2}, g(x) = \sqrt{1+x}$ in $\left[0, \frac{1}{2}\right]$

2. If f' and g' are continuous and differentiable on $[a, b]$, then show that $a < c < b$

$$\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}.$$

3. Show that $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, 0 < \alpha < \theta < \beta < \frac{\pi}{2}$.

Answers

1. a. $c = -\frac{\pi}{4}$ b. $c = 14/9$ c. $c = \sqrt{3}$ d. $c = \frac{e}{e-1}$

e. $c = \frac{\sqrt{6}-1}{\sqrt{6}}$

INTERESTING FACTS

- Rolle's Theorem establishes a connection between continuity and differentiability.
- Mean value theorem is even used to check the accuracy of a speedometer.
- It specifies the existence of a point where the derivative vanishes.

VIDEO REFERENCES

Rolle's Theorem

Rolle's Theorem & Lagrange Mean Value Theorem (MVT)

Mean Value Theorems

USES OF ICT

- https://www.mathwarehouse.com/calculus/derivatives/what-is-rolles-theorem.php

APPLICATIONS TO REAL LIFE

- If the average speed during a journey from A to B was say 50 kms/hour, then there had to be a time when the instantaneous speed was 50 kms/hour as well (that is the maximum)
- The rate of change in timings of the sunset, over the seasons.
- When a ball is thrown upwards in the air, its velocity becomes zero at some point of time. Rolle's Theorem explains that the velocity of ball becomes zero at some point of time.
- LMVT is used to issue "*challan*" for speeding.

2.2 TAYLOR'S THEOREM

Taylor's theorem is an extension of mean value theorem as mean value theorem relates the value of function and its first order derivative but Taylor's theorem relates the value of function and its 'higher order derivatives'.

2.2.1 Taylor's Theorem with Lagrange's form of Remainder

Statement: If a function $f : [a, a + h] \to \mathbb{R}$ be such that

i. $f, f', f'', ..., f^{n-1}$ are continuous function of x in the closed interval $[a, a + h]$.

ii. $f^{n}(x)$ exists in the open interval $(a, a + h)$ then there exists atleast one real number θ; $0 < \theta < 1$, such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(\theta h + a)$$

Proof: Consider a function $\phi : [a, a + h] \to \mathbb{R}$ in such a way that

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ... \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{n!} A \qquad ...(1)$$

where A is a constant to be chosen such that

$$\phi(a) = \phi(a + h) \qquad ...(2)$$

Now putting $x = a$ and $x = a + h$ in (1), we have

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A$$

and $\qquad \phi(a + h) = f(a + h)$

Putting these values in (2), we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \qquad ...(3)$$

Now,

i. $\phi(x)$ is continuous on $[a, a+h]$, since $f(x), f'(x), f''(x), ..., f^{n-1}(x)$ are continuous on $[a, a+h]$ and $(a+h-x), (a+h-x)^2, ..., (a+h-x)^n$ being polynomials are also continuous on closed interval $[a, a+h]$. Also the algebraic sum of continuous functions is continuous.

ii. $\phi(x)$ is derivable in $(a, a + h)$, since $f(x), f'(x), ... + f^{n-1}(x)$ are all derivable on $(a, a + h)$. Also $(a + h - x), (a + h - x)^2, ... (a + h - x)^n$ being polynomials are derivable in the open interval $(a, a + h)$.

iii Also, $\phi(a) = \phi(a + h)$,

$\therefore$ $\phi(x)$ satisfies all the three conditions of Rolle's theorem in $[a, a + h]$. Hence there exists atleast one real number θ, $0 < \theta < 1$, such that

$$\phi'(a + \theta h) = 0 \qquad ...(4)$$

Differentiating (1) w.r.t. x, we get

$$\phi'(x) = f'(x) + (a+h-x) f''(x) - f'(x) + \frac{(a+h-x)^2}{2!} f'''(x) + \frac{2(a+h-x)}{2!} (-1) f''(x)$$

$$+ ... + ... + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(n-1)(a+h-x)^{n-2}}{(n-1)!} f^{n-1}(x)$$

$$+ \frac{n(a+h-x)^{n-1}(-1)}{n!} A$$

or
$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^n(x) - A]$$

Putting $x = a + \theta h$

$$\phi'(a + \theta h) = \frac{[h(1-\theta)]^{n-1}}{(n-1)!} [f^n(a + \theta h) - A]$$

But $\qquad \phi'(a + \theta h) = 0 \qquad$ [From 4]

$\Rightarrow \qquad f^n(a + \theta h) - A = 0 \quad \Rightarrow \quad A = f^n(a + \theta h) \qquad [\because \quad 1 - \theta \neq 0$ and $h \neq 0]$

From (3),
$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

which is the required result of the theorem.

Here, the $(n + 1)^{\text{th}}$ term *i.e.*, $\dfrac{h^n}{n!} f^n(a + \theta h)$ is called the Lagrange's form of remainder after nth term.

2.2.2 Maclaurin's Theorem with Lagrange's Form of Remainder

Statement: If a function $f(x)$ defined in closed interval $[0, x]$ is such that

i. $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continuous in closed interval $[0, x]$

ii. $f^n(x)$ exists in open interval $(0, x)$, then there exist atleast one real number $\theta, 0 < \theta < 1$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

Proof: From Taylor's theorem, we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

In this expression, put $a = 0$, $h = x$, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x), 0 < \theta < 1$$

The above expression is the required Maclaurin's theorem with Lagrange's form of remainder.

2.2.3 Taylor's Theorem with Cauchy's Form of Remainder

Statement: If a function $f : [a, a + h] \to \mathbb{R}$ be such that

i. $f, f', f'', \ldots, f^{n-1}$ are all continuous function of x in the closed interval $[a, a + h]$

ii. $f^n(x)$ exists in the open interval $(a, a + h)$, then there exists atleast one real number θ; $0 < \theta < 1$, such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a + \theta h)$$

Proof: Consider a function $\phi : [a, a + h] \to \mathbb{R}$ by

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a + h - x) . A \quad \ldots(1), x \in (a, a + h)$$

where A is constant to be chosen such that $\phi(a) = \phi(a + h)$

Putting $x = a$ in (1), $\quad \phi(a) = f(a) + hf'(a) + \dfrac{h^2}{2!} f''(a) + \ldots + \dfrac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \quad \ldots(2)$

and $\quad \phi(a + h) = f(a + h)$

Putting $x = a + h$, in (1), we have

$$\phi(a + h) = f(a + h) + 0 + 0 + \ldots + 0 = f(a + h) \quad \ldots(3)$$

Now, $\quad \phi(a + h) = \phi(a)$

Then, from (2) and (3), we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hA \quad \ldots(4)$$

Now,

i. ϕ is continuous on $[a, a+h]$, since $f, f', f'', \ldots f^{n-1}$ are continuous on $[a, a+h]$ and also $(a+h-x), (a+h-x)^2, \ldots, (a+h-x)^{n-1}$ being polynomials are continuous on $[a, a+h]$

ii. ϕ is differentiable in $[a, a+h]$, since $f, f', f'', \ldots f^{n-1}$ are all differentiable in $(a, a+h)$ and also $(a+h-x), (a+h-x)^2, \ldots, (a+h-x)^{n-1}$ being polynomial are derivable in $(a, a+h)$

iii. Also $\phi(a) = \phi(a+h)$

Now, $\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exist atleast one real number θ, $0 < \theta < 1$, such that $\phi'(a+\theta h) = 0$

Differentiating both sides of (1) w.r.t. x, we have,

$$\phi'(x) = f'(x) + [(a+h-x)f''(x) - f'(x)] + \frac{1}{2!}[2(a+h-x)f''(x)(-1)$$

$$\frac{(a+h-x)^2}{2!} f'''(x)] + \ldots + \ldots + \frac{1}{(n-1)!}[(a+h-x)^{n-1} f^n(x)$$

$$-(n-1)(a+h-x)^{n-2} f^{n-1}(x)] - A$$

or $$\phi'(x) = \frac{(a+h-x)^{n-1} f^n(x)}{(n-1)!} - A$$

Putting $x = a + \theta h$, we have

$$\phi'(a+\theta h) = \frac{(a+h-a-\theta h)^{n-1}\, f^n(a+\theta h)}{(n-1)!} - A$$

$$= \frac{[h(1-\theta)]^{n-1}\, f^n(a+\theta h)}{(n-1)!} - A$$

$$= \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h) - A$$

But $$\phi'(a+\theta h) = 0$$

$$\Rightarrow \quad \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h) - A = 0$$

or $$A = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$$

Putting this value of 'A' in (4), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + h\left[\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)\right]$$

i.e., $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$$

is the required form of Taylor's theorem with Cauchy's form of remainder.

Here, $\frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$ is called the Cauchy form of remainder after n^{th} term.

2.2.4 Maclaurin's Theorem with Cauchy's Form of Remainder

Statement: If a function $f(x)$ defined on $[0, x]$ is such that

i. $f(x), f'(x), f''(x), ..., f^{n-1}(x)$ are continuous function in $[0, x]$

ii. $f^n(x)$ exists in $(0, x)$,

then there exist atleast one real number θ, $0 < \theta < 1$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), 0 < \theta < 1$$

Proof: From Taylor's theorem with Cauchy form of remainder, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h)$$

Put $a = 0$ and $h = x$ in above theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^n(\theta x)$$

which is the Maclaurin's theorem with Cauchy's form of remainder.

SOME SOLVED EXAMPLES

Example 2.16. *The expansion of the function $f(x) = (1-x)^{7/2}$ is given by $f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$. Find the value of θ as $x \to 1$.*

Solution. Here,

$$f(x) = (1-x)^{7/2}$$

$$f'(x) = -\frac{7}{2}(1-x)^{5/2}$$

$$f''(x) = \frac{35}{4}(1-x)^{3/2}$$

$$f'''(x) = -\frac{105}{8}(1-x)^{1/2}$$

Finding all above derivatives at $x = 0$, we have

$$f(0) = 1, f'(0) = \frac{-7}{2}, f''(0) = \frac{35}{4}$$

$$f'''(\theta x) = -\frac{105}{8}(1-\theta x)^{1/2}$$

Now,
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$$

$$\Rightarrow \quad (1-x)^{7/2} = 1 - \frac{7}{2}x + \frac{35}{8}x^2 - \frac{105}{48}(1-\theta x)^{1/2} . x^3$$

When $x \to 1$, we have
$$0 = 1 - \frac{7}{2} + \frac{35}{8} - \frac{105}{48}(1-\theta)^{1/2}$$

$$\Rightarrow \qquad \frac{105}{48}\,(1-\theta)^{1/2} = \frac{15}{8}$$

$$\Rightarrow \qquad (1-\theta)^{1/2} = \frac{15}{8}\times\frac{48}{105}$$

$$\Rightarrow \qquad (1-\theta)^{1/2} = \frac{6}{7}$$

Squaring on both sides

$$1-\theta = \frac{36}{49}$$

$$\Rightarrow \qquad \theta = 1-\frac{36}{49}$$

$$\Rightarrow \qquad \theta = \frac{13}{49} \quad \textbf{Answer}$$

Example 2.17. *Show that for every value of x,*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \;...+ (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin(\theta x).$$

Solution. Given, $f(x) = \cos x$

$$\therefore \qquad f'(x) = -\sin x = \cos\left(x+\frac{\pi}{2}\right)$$

$$f''(x) = -\cos x = \cos(x+\pi)$$

$$f'''(x) = \sin x = \cos\left(\frac{3\pi}{2}+x\right)$$

$$f^{iv}(x) = \cos x = \cos(2\pi + x)$$

....

....

$$f^{n}(x) = \cos\left[x+\frac{n\pi}{2}\right]$$

$$f^{2n-1}(x) = \cos\left[x+(2n-1)\frac{\pi}{2}\right]$$

$$f^{2n}(x) = \cos\left[x+2n\,.\,\frac{\pi}{2}\right]$$

$$= \cos[x+n\pi]$$

$$f^{2n+1}(x) = \cos\left[x+(2n+1)\frac{\pi}{2}\right]$$

$$f^{2n+1}(\theta x) = \cos\left[\theta x+(2n+1)\frac{\pi}{2}\right]$$

So, $f(0) = \cos 0 = 1, \qquad f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1, \qquad f'''(0) = \sin 0 = 0$

$f^{iv}(0) = \cos 0 = 1$

....

....

$$f^{2n-1}(0) = \cos\left[(2n-1)\frac{\pi}{2}\right]$$

$$= \cos\left(n\pi - \frac{\pi}{2}\right) = 0$$

$$f^{2n}(0) = \cos n\pi = \begin{cases} 1, & n = \text{even} \\ -1, & n = \text{odd} \end{cases} = (-1)^n$$

$$f^{2n+1}(\theta x) = \cos\left[\theta x + n\pi + \frac{\pi}{2}\right]$$

$$= \begin{cases} -\sin\theta x, & n = \text{even} \\ \sin\theta x, & n = \text{odd} \end{cases} = (-1)^{n+1}\sin\theta x$$

By Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^{2n}}{(2n)!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

On substituting the values, we have

$$\cos x = 1 + x\,.\,0 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \ldots + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^{n+1}\sin\theta x$$

i.e., $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin\theta x$$ **Proved.**

Example 2.18. *If a function f is such that f' is continuous on [a, b] and derivable on (a, b). Show that there exist a real number θ, 0 < θ < 1, such that*

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''[a + \theta(b-a)].$$

Solution. Consider the function

$$\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 A \quad \ldots(1)$$

where A is constant to be chosen such that

$$\phi(a) = \phi(b)$$

Now, $$\phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 A \quad \ldots(2)$$

and $$\phi(b) = f(b) \quad \text{[obtained by putting } x = a \text{ and } x = b \text{ in (1)]} \ldots(3)$$

Using (3) in (2), we have

$$f(x) = f(a) + (b-a)f'(a) + (b-a)^2 A \quad \ldots(4)$$

i. As $f(x), f'(x)$, are continuous on $[a, b]$ and $(b-x)$, $(b-x)^2$ being polynomial are also continuous on $[a, b]$

∴ $\phi(x)$ is continuous on $[a, b]$

ii. As $f(x), f'(x)$, are derivable on (a, b) and $(b-x)$, $(b-x)^2$ being polynomial are also derivable on (a, b)

∴ $\phi(x)$ is derivable on (a, b)

iii. Also $\phi(a) = \phi(b)$

Thus, $\phi(x)$ satisfies all three conditions of Rolle's theorem.

Hence there exists a real number θ, $0 < \theta < 1$ such that $\phi'[a + \theta(b - a)] = 0$ ...(5)

Differentiating (1) w.r.t. x, we have

$$\phi'(x) = f'(x) + (b - x) f''(x) + (-1) f'(x) + 2(b - x)(-1) A$$

or $$\phi'(x) = (b - x) f''(x) - 2(b - x) A$$

or $$\phi'(x) = (b - x)[f''(x) - 2A]$$

Putting $x = a + \theta(b - a)$, we have

$$\phi'[a + \theta(b - a)] = [b - a - \theta(b - a)][f''(a + \theta(b - a) - 2A]$$

$$\Rightarrow \quad 0 = (b - a)(1 - \theta)[f''(a + \theta(b - a) - 2A]$$

[from (5) and as $b - a \neq 0$, $1 - \theta \neq 0$]

$$\Rightarrow \quad A = \frac{1}{2} f''(a + \theta(b - a))$$

Putting the value of 'A' in (4), we have

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b - a)^2}{2!} f''(a + \theta(b - a))$$

Example 2.19. *Show that* $\log(x + h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \ldots + (-1)^{n-1} \frac{h^n}{n(x + \theta h)^n}$

Solution. Let $f(x + h) = \log(x + h)$

$$\therefore \quad f(x) = \log x$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f''' = \frac{2}{x^3}$$

$$f^{iv}(x) = \frac{-6}{x^4} = (-1)^3 \cdot \frac{3!}{x^4}$$

....

....

Continuing like this

$$f^{n-1}(x) = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$f^n(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

$$f^n(x + \theta h) = \frac{(-1)^{n-1}(n-1)!}{(x + \theta h)^n}$$

By Talylor's theorem with Lagrange's form of remainder, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x+\theta h)$$

After putting all values, we have

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3!} \times \frac{2!}{x^3} + \ldots + \frac{h^{n-1}}{(n-1)!} \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \frac{h^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(x+\theta h)^n}$$

or $$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + \ldots + \frac{h^{n-1}(-1)^{n-2}}{(n-1)x^{n-1}} + \frac{(-1)^{n-1} h^n}{n(x+\theta h)^n}$$

Example 2.20. *Expand $e^{ax} \sin bx$ by Maclaurin's theorem with Lagrange's form of remainder after n terms.*

Solution. Let $f(x) = e^{ax} \sin bx$

$$\therefore \quad f'(x) = e^{ax} \cos bx \,.\, b + ae^{ax} \sin bx$$

$$= e^{ax}(b \cos bx + a \sin bx)$$

$$f''(x) = e^{ax}(-b^2 \sin bx + ab \cos bx) + ae^{ax}(b \cos bx + a \sin bx)$$

$$= e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$$

$$f'''(x) = e^{ax}[(a^2 - b^2) \cos bx \,.\, b - 2ab \sin bx \,.\, b + ae^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$$

$$= e^{ax}[b(3a^2 - b^2) \cos bx + (a^3 - 3ab^2) \sin bx]$$

Continuing like this, we have

$$f^n(x) = (a^2 + b^2)^{n/2} e^{ax} \sin\left(bx + n \tan^{-1} \frac{b}{a}\right)$$

At $x = 0$, we have

$$f(0) = 0, f'(0) = b, \qquad f'(0) = 2ab,$$

$$f'''(0) = b(3a^2 - b^2)$$

$$f^n(\theta x) = (a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1} \frac{b}{a}\right)$$

According to Maclaurin's theorem with Lagrange's form of remainder, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

After putting all values, we have

$$e^{ax} \sin bx = 0 + x \,.\, b + \frac{x^2}{2!}(2ab) + \ldots + \frac{x^n}{n!}(a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1} \frac{b}{a}\right)$$

or $$e^{ax} \sin bx = bx + \frac{x^2}{2!}(2ab) + \frac{x^3}{3!} b(3a^2 - b^2) + \ldots$$

$$+ \frac{x^n}{n!}(a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1} \frac{b}{a}\right)$$

EXERCISE 2.4

1. Show that for every value of x, the expansion,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!} \sin \theta x, \; 0 < \theta < 1.$$

2. With the help of Maclaurin's expansion, show that

   a. $\log (1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots + (-1)^{n-2} \dfrac{x^{n-1}}{n-1} + (-1)^{n-1} \dfrac{x^n}{n(1+\theta x)^n}$.

   b. $\log (1 - x) = - x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dots - \dfrac{x^{n-1}}{n-1} - \dfrac{x^n}{n(1-\theta x)^n}$.

3. If f' is continuous on $[a, a + h]$ and derivable on $(a, a + h)$, then prove that there exist a real number c between a and $(a + h)$, such that $f(a + h) = f(a) + hf'(a) + \dfrac{h^2}{2!} f''(c)$.

4. The expansion of a function $f(x) = (1 - x)^{5/2}$ is given by $f(0) + xf'(0) + \dfrac{x^2}{2!} f''(\theta x)$. Find the value of θ as $x \to 1$.

5. Expand $\sqrt{x}$ in ascending power of x by using Maclaurin's theorem, if possible.

6. Expand the function $f(x) = a^x$ by using Maclaurin's theorem with Lagranges form of remainder after n terms.

7. Expand $e^{ax} \sin bx$ by using Maclaurin's theorem with Cauchy's form of remainder after n terms.

Answers

4. $\theta = \dfrac{9}{25}$

6. $1 + x \log a + \dfrac{x^2}{2!} (\log a)^2 + \dots + \dfrac{x^{n-1}}{(n-1)!} (\log a)^{n-1} + \dfrac{x^n}{n!} a^{\theta x} (\log a)^n$

7. $bx + (2ab) \dfrac{x^2}{2!} + b(3a^2 - b^2) \dfrac{x^3}{3!} + \dots + (a^2 + b^2)^{n/2} (1 - \theta)^{n-1} \dfrac{x^n}{(n-1)!} e^{a\theta x} \sin \left(b\theta x + n \tan^{-1} \dfrac{b}{a} \right)$

INTERESTING FACTS

- It is even used in signal processing industry where we need to approximate sinusoidal functions.
- It is used in transistors and amplifiers industry to check the effect of signal.

VIDEO REFERENCES

Taylor's Theorem 1

Taylor's Theorem 2

APPLICATIONS TO REAL LIFE

- These help in calculating the approximate values of many functions on computers and calculators.
- They are very useful in solving the limits and determining several infinite sums.
- These are very helpful in understanding the asymptotic behaviour of functions.

2.3 INDETERMINATE FORMS AND L′HOSPITAL'S RULE

Let $f(x)$ and $g(x)$ be the given two functions. Then the limit of $f(x)/g(x)$ as $x \to c$ is, in general, equal to the limit of the numerator divided by the limit of the denominator. But when those two limits are both zero, the quotient reduces to the form 0/0.

The form 0/0 is called an indeterminate form.

Mathematically, it can be expressed as,

For evaluating the limit, $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{l}{m}$

if $l = 0$, $m \neq 0$, then $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)} = 0$

if $l \neq 0$, $m = 0$, then $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)} = \infty$

if $l = 0$, $m = 0$, then $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{0}$

cannot be evaluated and this is called indeterminate form.

Different indeterminate forms are represented by symbols as follows

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$$

Here, we will explain all these indeterminate forms with examples.

2.3.1 L'Hospital Rule for Evaluation of Indeterminate form $\frac{0}{0}$ (Type-I)

Theorem: Let the functions f and g are differentiable function at $x = a$ and $f(a) = 0 = g(a)$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Proof: As $f(a) = 0 = g(a)$

We can write, $\displaystyle\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)}$

Dividing by $x - a$, we have

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

Taking limit on both sides, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} \qquad \left[\because \lim_{x\to b}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to b} f(x)}{\lim\limits_{x\to a} g(x)}\right]$$

$$= \frac{f'(a)}{g'(a)} \qquad \text{[As per definition of differentiability]}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Generally, if $f(a) = f'(a) = f''(a) \ldots = f^{n-1}(a) = 0$

$g(a) = g'(a) = g''(a) \ldots = g^{n-1}(a) = 0$

and $g^n(a) \neq 0$

then, if $\lim\limits_{x\to a}\dfrac{f^n(x)}{g^n(x)}$ exists,

then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^n(x)}{g^n(x)}$$

This is known as L'Hospital's Rule.

Working Rule

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is undefined and is of form $\dfrac{0}{0}$, then evaluating the limit by following procedures:

1. Differentiate the numerator and denominator separately *i.e.*, apply L'Hospital's rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Two cases arise:

Case I: If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ is not of the form $\dfrac{0}{0}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Case II: If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ is of the form $\dfrac{0}{0}$, then again differentiate numerator and denominator separately *i.e.*, apply L'Hospital Rule such that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$$

2. Repeat the above procedure (Case-II) till $\lim\limits_{x \to a} \dfrac{f^n(x)}{g^n(x)}$ reach to determinate form, *i.e.*,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}$$

SOME SOLVED EXAMPLES (TYPE-I)

Example 2.21. *Evaluate* $\lim\limits_{x \to 0} \dfrac{1-\cos x}{3x^2}$.

Solution. Given, $\lim\limits_{x \to 0} \dfrac{1-\cos x}{3x^2}$ $\left[\dfrac{0}{0} \text{ form}\right]$

Apply L'Hospital Rule, $= \lim\limits_{x \to 0} \dfrac{\sin x}{6x}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{\cos x}{6}$$

$$= \frac{1}{6}$$

Example 2.22. *Evaluate* $\lim\limits_{x \to 1} \dfrac{x^x - x}{x - 1 - \log x}$.

Solution. Given, $\lim\limits_{x \to 1} \dfrac{x^x - x}{x - 1 - \log x}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 1} \frac{x^x(1+\log x) - 1}{1 - \frac{1}{x}} \quad \text{(Apply L'Hospital Rule)}$$

$$= \lim_{x \to 1} \frac{x^x + x^x \log x - 1}{1 - \frac{1}{x}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{x^x(1+\log x) + x^x \cdot \frac{1}{x} + x^x(1+\log x)\log x}{\frac{1}{x^2}}$$

$$= \lim_{x \to 1} \frac{x^x + x^x \log x + x^{x-1} + (x^x + x^x \log x)\log x}{\frac{1}{x^2}}$$

$$= \frac{1+0+1+0}{1} = 2.$$

Example 2.23. *Find the value of 'a' and 'b' so that* $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{x^3}=1$.

Solution. Given, $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{x^3} = 1$ $\qquad\left[\dfrac{0}{0}\text{ form}\right]$

$$\Rightarrow \quad \lim_{x\to 0}\frac{x(-a\sin x)+(1+a\cos x)-b\cos x}{3x^2} = 1 \qquad \text{...(1)}$$

Since R.H.S. of (1) is finite, so L.H.S. must finite, when $x \to 0$

But denominator $\to 0$ as $x \to 0$

and numerator $\to 0$ as $x \to 0$

$$\Rightarrow \quad 1 + a - b = 0$$

$$\Rightarrow \quad a - b = -1 \qquad \text{...(2)}$$

Again Applying L'Hospital Rule on L.H.S. of (1), we have

$$\lim_{x\to 0}\frac{-a\sin x - ax\cos x - a\sin x + b\sin x}{6x} = 1 \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$\lim_{x\to 0}\frac{-a\cos x - a\cos x + ax\sin x - a\cos x + b\cos x}{6} = 1$$

$$\Rightarrow \quad \frac{-3a+b}{6} = 1 \quad \Rightarrow \quad -3a + b = 6 \qquad \text{...(3)}$$

Using (2) and (3), we get, $a = -5/2$, $b = -3/2$ **Answer**

Example 2.24. *Evaluate* $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x - x^2}{x^3}$.

Solution. $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x - x^2}{x^3}$ $\qquad\left[\dfrac{0}{0}\text{ form}\right]$

Applying L'Hospital Rule,

$$\lim_{x\to 0}\frac{e^x\cos x + e^x\sin x - 1 - 2x}{3x^2}$$

or

$$\lim_{x\to 0}\frac{e^x(\cos x+\sin x)-1-2x}{3x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{e^x[\cos x+\sin x]+e^x[-\sin x+\cos x]-2}{6x}$$

$$= \lim_{x\to 0}\frac{2e^x\cos x - 2}{6x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2e^x\cos x - 2e^x\sin x}{6} = \frac{2}{6} = \frac{1}{3}$$ **Answer**

Example 2.25. *Find 'a' such that* $\lim\limits_{x\to 0}\dfrac{a\sin x - \sin 2x}{\tan^3 x}$ *is finite.*

Solution. $\lim\limits_{x\to 0}\dfrac{a\sin x - \sin 2x}{\tan^3 x}$ $\qquad\left[\dfrac{0}{0}\text{ form}\right]$

Applying L'Hospital Rule, we have

$$\lim_{x\to 0} \frac{a\cos x - 2\cos 2x}{3\tan^2 x \,.\, \sec^2 x} = \frac{a-2}{0}$$

But it is given that, $\lim_{x\to 0} \dfrac{a\sin x - \sin 2x}{\tan^3 x}$ is finite

$\therefore \qquad a - 2 = 0$

$\Rightarrow \qquad a = 2$ **Answer**

2.3.1.1 Evaluation of Limit by Method of Expansion of Series

i. $a^x = 1 + x\log a + \dfrac{x^2}{2!}(\log a)^2 + \dfrac{x^3}{3!}(\log a)^3 + \ldots\ldots$

ii. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots$

iii. $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots, |x| < 1$

iv. $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} \ldots\ldots, |x| < 1$

v. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots, |x| < 1$

vi. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots\ldots, |x| < 1$

vii. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\ldots, \qquad \forall x$

viii. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots, \qquad \forall x$

ix. $\sin hx = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\ldots, \qquad \forall x$

x. $\cos hx = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\ldots, \qquad \forall x$

xi. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots\ldots, \qquad \forall x$

xii. $(1+x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$

Example 2.26. *Evaluate* $\lim_{x\to 0} \dfrac{1-\cos x}{x\log(1+x)}$.

Solution. Here $\lim_{x\to 0} \dfrac{1-\cos x}{x\log(1+x)}$ $\qquad \left[\dfrac{0}{0}\text{ form}\right]$

$$= \lim_{x \to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} ......\right)}{x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right)}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + ...}{x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + ...}$$

$$= \lim_{x \to 0} \frac{x^2\left(\frac{1}{2!} - \frac{x^2}{4!} + ...\right)}{x^2\left(1 - \frac{x}{2} + \frac{x^2}{3} ...\right)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} - \frac{x^2}{4!} + ...}{1 - \frac{x}{2} + \frac{x^2}{3} ...}$$

$$= \frac{\frac{1}{2}}{1} = \frac{1}{2} \quad \textbf{Answer}$$

Alternate Method: $\lim_{x \to 0} \frac{1 - \cos x}{x \log(1+x)}$ $\left[\frac{0}{0} \text{ form}\right]$

Apply L'Hospital Rule, $\lim_{x \to 0} \frac{\sin x}{\frac{x}{1+x} + \log(1+x)}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{(1+x)^2} + \frac{1}{1+x}} = \frac{1}{1+1} = \frac{1}{2}.$$

Example 2.27. *Evaluate* $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$.

Solution. Given, $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{x\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} ...\right] - \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right]}{x^2}$$

$$= \lim_{x \to 0} \frac{\left[x - \frac{x^3}{2!} + \frac{x^5}{4!} ...\right] - \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right]}{x^2}$$

$$= \lim_{x \to 0} \frac{\left[\frac{x^2}{2} - \frac{5x^3}{6} + \ldots\right]}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{1}{2} - \frac{5x}{6}\right] = \frac{1}{2} \quad \textbf{Answer}$$

EXERCISE 2.5

1. Evaluate the following:

a. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

b. $\lim_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin x^2}$

c. $\lim_{x \to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

d. $\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)}$

e. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^2 \sin x}$

f. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

g. $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$

h. $\lim_{x \to 0} \frac{3^x - 2^x}{\sqrt{x}}$

i. $\lim_{x \to 0} \frac{\sin hx - x}{\sin x - x \cos x}$

j. $\lim_{x \to 0} \frac{x - \sin x}{e^{\sin x} - e^x}$

2. Evaluate the following limits:

a. $\lim_{a \to b} \frac{a^b - b^a}{a^a - b^b}$

b. $\lim_{x \to 0} \frac{\cos hx - \cos x}{x \sin x}$

c. $\lim_{x \to b} \frac{x^b - b^x}{x^x - b^b}$

d. $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$

e. $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$

f. $\lim_{x \to \pi/2} \frac{\left(\frac{\pi}{2} - x\right)^2 \sin x}{\cos^2 x}$

g. $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

h. $\lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x}$

3. If $\lim_{x \to 0} \frac{re^x - q\cos x + pe^{-x}}{x \tan x} = 3$, find the values of p, q and r.

4. Find the value of a, b, c so that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$.

5. Evaluate:

i. $\lim_{x \to 0} \frac{\sin x}{x}$

ii. $\lim_{x \to 0} \frac{\tan x}{x}$

Answers

1.	a. 1	b. $\frac{1}{2}$	c. $-\frac{1}{2}$	d. 1
	e. $\frac{1}{3}$	f. 1	g. $\frac{\log a}{\log b}$	h. 0
	i. $\frac{1}{2}$	j. -1		
2.	a. $\frac{1-\log b}{1+\log b}$	b. $\frac{1}{2}$	c. $\frac{1-\log b}{1+\log b}$	d. $\frac{1}{3}$
	e. $3/2$	f. 1	g. $-2/3$	h. 2
3.	$p=\frac{3}{2}, q=3, r=\frac{3}{2}$	**4.** $a=1, b=2, c=1$	**5.** i. 1	ii. 1

2.3.2 L'Hospital Rule for Evaluation of Indeterminate Form $\frac{\infty}{\infty}$ (Type-II)

Theorem: If $\lim\limits_{x\to a} f(x)=\infty$ and $\lim\limits_{x\to a} g(x)=\infty$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ provided that R.H.S. exists (whether finite or infinite).

Working Rule:

1. For evaluating the indeterminate form $\frac{\infty}{\infty}$, change them to the form $\frac{0}{0}$ and then solve.
2. When $x\to\infty$, then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}$ when $x\to\infty$, change $x\to\frac{1}{y}$ so that $y\to 0$

   Let $x=\frac{1}{y}$ then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{y\to 0}\frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)}$ and proceed further.

SOME SOLVED EXAMPLES (TYPE-II)

Example 2.28. *Evaluate* $\lim\limits_{x\to 0^+}\frac{\log \tan 2x}{\log \tan x}$.

Solution. Given, $\lim\limits_{x\to 0^+}\frac{\log\tan 2x}{\log\tan x}$ $\left[\frac{\infty}{\infty}\text{ form}\right]$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\tan 2x}\times\sec^2 2x\times 2}{\frac{1}{\tan x}\times\sec^2 x}$$

$$= \lim_{x \to 0^+} \frac{2 \tan x \, . \sec^2 2x}{\tan 2x \, . \sec^2 x}$$

$$= \lim_{x \to 0^+} \frac{2 \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos^2 2x}}{\dfrac{\sin 2x}{\cos 2x} \times \dfrac{1}{\cos^2 x}}$$

$$= \lim_{x \to 0^+} \frac{2 \sin x}{\sin 2x} \frac{\cos x}{\cos 2x}$$

$$= \lim_{x \to 0^+} \frac{\sin 2x}{\sin 2x \cos 2x} = \lim_{x \to 0^+} \frac{1}{\cos 2x}$$

$$= 1$$

Example 2.29. *Evaluate* $\lim\limits_{x \to \infty} \dfrac{2x^4 + 3x^3 - 100}{4x^4 + x^2 + 2x + 100}$.

Solution. Given $\lim\limits_{x \to \infty} \dfrac{2x^4 + 3x^3 - 100}{4x^4 + x^2 + 2x + 100}$

Put $x = \dfrac{1}{y}$ as $x \to \infty \quad \Rightarrow \quad y \to 0$

$$\therefore \qquad \lim_{y \to 0} \frac{2 \, . \dfrac{1}{y^4} + 3 \, . \dfrac{1}{y^3} - 100}{4 \, . \dfrac{1}{y^4} + \dfrac{1}{y^2} + 2 \, . \dfrac{1}{y} + 100}$$

$$= \lim_{y \to 0} \frac{2 + 3y - 100y^4}{4 + y^2 + 2y^3 + 100y^4} = \frac{2}{4} = \frac{1}{2}$$

Example 2.30. *Evaluate* $\lim\limits_{x \to \pi/2} \dfrac{\tan x}{\tan 3x}$.

Solution. Here $\lim\limits_{x \to \pi/2} \dfrac{\tan x}{\tan 3x}$ $\qquad \left[\dfrac{\infty}{\infty} \text{ form}\right]$

$$\Rightarrow \qquad \lim_{x \to \pi/2} \frac{\sec^2 x}{3 \sec^2 3x}$$

$$= \lim_{x \to \pi/2} \frac{1}{3} \frac{\cos^2 3x}{\cos^2 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{1}{3} \left[\frac{-2 \cos 3x \, . \sin 3x \, . 3}{-2 \cos x \, . \sin x}\right]$$

$$= \lim_{x \to \pi/2} \frac{\sin 6x}{\sin 2x} \qquad [\because \quad \sin 2x = 2 \sin x \cos x] \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{6 \cos 6x}{2 \cos 2x} = \frac{3 \cos 3\pi}{\cos \pi} = \frac{3(-1)}{(-1)} = 3.$$

2.3.3 L' Hospital Rule for Evaluation of Indeterminate Form 0 × ∞ (Type-III)

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} f(x) \,.\, g(x)$ is of the form $0 \times \infty$

Converting,

$$\lim_{x \to a} f(x) \,.\, g(x) = \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}} \text{ or } \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}}$$

which is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively and then solve with previous discussed methods.

SOME SOLVED EXAMPLES (TYPE-III)

Example 2.31. *Evaluate* $\lim_{x \to 0^+} x \log x$.

Solution. Here, $\lim_{x \to 0^+} x \log x$ $\quad [0 \times \infty \text{ form}]$

$$= \lim_{x \to 0^+} \frac{\log x}{1/x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hospital Rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

Example 2.32. *Evaluate* $\lim_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right)$.

Solution. Given $\lim_{x \to 0} x \cot x$ $\quad [0 \times \infty \text{ form}]$

$$= \lim_{x \to 0} \frac{x}{\frac{1}{\cot x}} = \lim_{x \to 0} \frac{x}{\tan x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{1}{\sec^2 x} = 1$$

Example 2.33. *Evaluate* $\lim_{x \to \frac{\pi^-}{2}} \left(x - \frac{\pi}{2}\right) \tan x$.

Solution. Given $\lim_{x \to \frac{\pi^-}{2}} \left(x - \frac{\pi}{2}\right) \tan x$ $\quad [0 \times \infty \text{ form}]$

$$= \lim_{x \to \frac{\pi^-}{2}} \frac{\left(x - \frac{\pi}{2}\right)}{\frac{1}{\tan x}} = \lim_{x \to \frac{\pi^-}{2}} \frac{\left(x - \frac{\pi}{2}\right)}{\cot x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Apply L'Hospital Rule,

$$\lim_{x \to \frac{\pi^-}{2}} \frac{1}{-\text{cosec}^2 x} = -1$$

Example 2.34. *Evaluate* $\lim_{x \to \infty} 2^x \sin \frac{a}{2^x}$.

Solution. Given, $\lim_{x \to \infty} 2^x \sin \frac{a}{2^x}$ $\qquad [0 \times \infty \text{ form}]$

$$\Rightarrow \qquad \lim_{x \to \infty} \frac{\sin \frac{a}{2^x}}{\frac{1}{2^x}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Applying L'Hospital Rule

$$\Rightarrow \qquad \lim_{x \to \infty} \frac{\cos \frac{a}{2x} \left(\frac{-a \,.\, 2^x \log 2}{(2^x)^2}\right)}{\frac{-2^x \log 2}{(2^x)^2}}$$

$$= \lim_{x \to \infty} a \cos \frac{a}{2^x} = a$$

2.3.4 L'Hospital Rule for Evaluation of the Indeterminate Form $\infty - \infty$ (Type-IV)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} (f(x) - g(x))$ is of the form $\infty - \infty$

In this form, convert,

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\, g(x)}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

which can be evaluated by L'Hospital Rule as did earlier.

SOME SOLVED EXAMPLES (TYPE-IV)

Example 2.35. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\text{cosec } x}{x}\right)$.

Solution. Given, $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\sin x}\right)$ [$\infty-\infty$ form]

$$= \lim_{x\to 0}\left(\frac{\sin x - x}{x^2 \sin x}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

Applying L'Hospital Rule

$$= \lim_{x\to 0}\left(\frac{\cos x - 1}{x^2 \cos x + 2x\sin x}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\sin x}{-x^2 \sin x + 2x\cos x + 2x\cos x + 2\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\sin x}{-x^2 \sin x + 4x\cos x + 2\sin x}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{-\cos x}{-x^2 \cos x - 6x\sin x + 6\cos x} = \frac{-1}{6}$$

Example 2.36. *Evaluate* $\lim_{x\to \pi/2}(\sec x - \tan x)$.

Solution. Here, $\lim_{x\to \pi/2}(\sec x - \tan x)$ [$\infty-\infty$ form]

$$= \lim_{x\to \pi/2}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to \pi/2}\left(\frac{1-\sin x}{\cos x}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to \pi/2}\left(\frac{-\cos x}{-\sin x}\right)$$

$$= \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0$$

Example 2.37. *Evaluate* $\lim_{x\to 0}\left(\operatorname{cosec}^2 x - \frac{1}{x^2}\right)$.

Solution. Given, $\lim_{x\to 0}\left(\operatorname{cosec}^2 x - \frac{1}{x^2}\right)$

or $\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$ [$\infty-\infty$ form]

$$= \lim_{x\to 0}\left(\frac{x^2 - \sin^2 x}{x^2 \sin^2 x}\right) \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right)\left(\frac{x^2}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right)\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right) \qquad \left(\because \lim_{x\to 0}\frac{\sin x}{x}=1\right)\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{x^2-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right)^2}{x^4}$$

$$= \lim_{x\to 0}\frac{x^2-x^2-\frac{x^4}{3}+\frac{x^6}{60}-\ldots}{x^4} = \lim_{x\to 0}\frac{\frac{x^4}{3}-\frac{x^6}{60}\ldots}{x^4}$$

$$= \lim_{x\to 0}\frac{1}{3}-\frac{x^2}{60} + \text{term containing higher power of } x$$

$$= \frac{1}{3}.$$

EXERCISE 2.6

1. Evaluate the following limits:

a. $\lim\limits_{x\to 0^+}\dfrac{\log\sin x}{\cot x}$

b. $\lim\limits_{x\to a^+}\dfrac{\log(x-a)}{\log(e^x-e^a)}$

c. $\lim\limits_{x\to\infty}\dfrac{x^3-8x^2+2x+1}{x^4-x^2+2x-3}$

d. $\lim\limits_{x\to\infty}\dfrac{\log x}{x}$

e. $\lim\limits_{x\to 0^+}\dfrac{\operatorname{cosec} x}{\log x}$

f. $\lim\limits_{x\to 0^+}\dfrac{\log x}{\cot x}$

g. $\lim\limits_{x\to\infty}\dfrac{x^n}{e^x},\ n\in N$

h. $\lim\limits_{x\to 5}\dfrac{\log(1-x)}{\cot(\pi x)}$

i. $\lim\limits_{x\to 0^+}\log_x \sin x$

2. Evaluate the following indeterminate forms:

a. $\lim\limits_{x\to 0} x\log x\tan x$

b. $\lim\limits_{x\to\infty} x\tan\dfrac{1}{x}$

c. $\lim\limits_{x\to a}(a-x)\tan\dfrac{\pi x}{2a}$

d. $\lim\limits_{x\to 1}\sec\dfrac{\pi}{2x}\log x$

e. $\lim\limits_{x\to\pi/2}(1-\sin x)\tan x$

f. $\lim\limits_{x\to\infty}(a^{1/x}-1)x$

3. Evaluate the following:

a. $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \operatorname{cosec} x\right)$

b. $\lim\limits_{x \to 0^+} \dfrac{\cot x - \dfrac{1}{x}}{x}$

c. $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \cot^2 x\right)$

d. $\lim\limits_{x \to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\log x}\right)$

e. $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x \tan x}\right)$

f. $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2} \log(1+x)\right)$

g. $\lim\limits_{x \to \pi/2} (2x \tan x - \pi \sec x)$

h. $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$

Answers

1. a. 0 b. 1 c. 0 d. 0
e. $-\infty$ f. 0 g. 0 h. 0
i. 1

2. a. 0 b. 1 c. $2a/\pi$ d. $2/\pi$
e. 0 f. $\log a$

3. a. 0 b. $-1/-3$ c. $\dfrac{2}{3}$ d. $\dfrac{1}{2}$
e. $\dfrac{1}{3}$ f. $\dfrac{1}{2}$ g. -2 h. 0

2.3.5 L'Hospital Rule for Evaluation of Indeterminate Form 0° (Type-V)

If $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} [f(x)]^{g(x)}$ is of the form 0°.

To solve this kind of form, let $y = \lim\limits_{x \to a} [f(x)]^{g(x)}$

$$\therefore \qquad \log y = \lim_{x \to a} g(x) \log f(x) \qquad \text{...(1)}$$

which is of the form $0 \times \infty$ and can be solved as previous method.

We can put,

$$\lim_{x \to a} g(x) \log f(x) = l,$$

then from (1), $\log y = l \quad \Rightarrow \quad y = e^l$.

SOME SOLVED EXAMPLES (TYPE-V)

Example 2.38. *Evaluate* $\lim\limits_{x \to 0^+} x^x$.

Solution. Let $y = \lim\limits_{x \to 0^+} x^x$ [0° form]

Taking log on both sides

$$\log y = \lim_{x \to 0^+} x \log x \qquad [0 \times \infty \text{ form}]$$

$$\log y = \lim_{x \to 0^+} \frac{\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hospital Rule,

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x \to 0^+} -x = 0$$

Thus $\qquad \log y = 0$

$\Rightarrow \qquad y = e^0 = 1$

Hence, $\qquad \lim_{x \to 0^+} x^x = 1.$

Example 2.39. *Evaluate* $\lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}}$.

Solution. Let

$$y = \lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}} \qquad [0^\circ \text{ form}]$$

then

$$\log y = \lim_{x \to 1} \frac{1}{\log(1-x)} \log(1 - x^2) \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{\log(1-x) + \log(1+x)}{\log(1-x)}$$

$$= \lim_{x \to 1} \left(1 + \frac{\log(1+x)}{\log(1-x)}\right)$$

$$= 1 + \lim_{x \to 1} \frac{\log(1+x)}{\log(1-x)}$$

$$= 1 + \lim_{x \to 1} \frac{\frac{1}{1+x}}{\frac{1}{1-x}} = 1 + \lim_{x \to 1} \frac{1-x}{1+x}$$

$$= 1 + \frac{0}{2} = 1$$

Thus, $\qquad \log y = 1$

$\Rightarrow \qquad y = e^1 = e$

Hence, $\lim_{x \to 1} (1 - x^2)^{\frac{1}{\log(1-x)}} = e.$

2.3.6 L'Hospital Rule for Evaluation of Indeterminate Form 1^∞ (Type-VI)

If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} [f(x)]^{g(x)}$ is of the form 1^∞.

It can be solve by taking $y = \lim_{x \to a} [f(x)]^{g(x)}$

$\therefore \quad \log y = \lim_{x \to a} g(x) \log f(x)$ ...(1) $[0 \times \infty$ form$]$

which can be evaluated as earlier method.

After that, let $\log y = l$ [From (1)]

then $y = e^l$

means $\lim_{x \to a} [f(x)]^{g(x)} = e^l$

SOME SOLVED EXAMPLES (TYPE-VI)

Example 2.40. *Evaluate* $\lim_{x \to 0} (\cos x)^{1/x^2}$.

Solution. Let $y = \lim_{x \to 0} (\cos x)^{1/x^2}$ $[1^\infty$ form$]$

$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x^2} \log \cos x$ $[0 \times \infty$ form$]$

Applying L'Hospital Rule

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x} \times (-\sin x)}{2x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-\sec^2 x}{2} = \frac{-1}{2}$$

Thus, $\log y = \frac{-1}{2}$

$\Rightarrow \quad y = e^{-1/2}$

Hence $\lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}$

Example 2.41. *Evaluate the given limit:* $\lim_{x \to 0} (\cos x)^{\cot^2 x}$.

Solution. Let $y = \lim_{x \to 0} (\cos x)^{\cot^2 x}$ $[1^\infty$ form$]$

$\therefore \quad \log y = \lim_{x \to 0} \cot^2 x \log \cos x$ $[0 \times \infty$ form$]$

$$= \lim_{x \to 0} \frac{\log \cos x}{\tan^2 x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Applying L'Hospital's Rule,

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x}.(-\sin x)}{2 \tan x . \sec^2 x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{-1}{2 \sec^2 x}$$

$$= -\frac{1}{2}$$

Thus, $\log y = -\frac{1}{2}$

$\Rightarrow$ $y = e^{-1/2}$

Hence, $\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$

2.3.7 L'Hospital Rule for Evaluation of Indeterminate form ∞^0 (Type-VII)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} [f(x)]^{g(x)}$ is of the form ∞^0.

For solving this kind of Indeterminate form,

Let $y = \lim_{x \to a} [f(x)]^{g(x)}$

$\therefore$ $\log y = \lim_{x \to a} g(x) \log f(x)$ ...(1) $[0 \times \infty \text{ form}]$

which is evaluated as previously discussed method

After that, let $\log y = l$ [from (1)]

then $y = e^l$

$\Rightarrow$ $\lim_{x \to a} [f(x)]^{g(x)} = e^l$

SOME SOLVED EXAMPLES (TYPE-VII)

Example 2.42. *Evaluate* $\lim_{x \to 0} (\cot x)^{1/\log x}$.

Solution. Let $y = \lim_{x \to 0} (\cot x)^{\frac{1}{\log x}}$ $[\infty^0 \text{ form}]$

$\therefore \qquad \log y = \lim_{x \to 0} \frac{1}{\log x} \log \cot x \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

Applying L'Hospital Rule

$$= \lim_{x \to 0} \frac{\frac{1}{\cot x} \times (-\text{cosec}^2 x)}{\frac{1}{x}}$$

$$= \lim_{x \to 0} \frac{-x \,\text{cosec}^2 x}{\cot x}$$

$$= \lim_{x \to 0} \frac{-x}{\cos x \sin x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{-1}{\cos^2 x - \sin^2 x} = -1$$

Thus, $\log y = -1 \quad \Rightarrow \quad y = e^{-1} = \frac{1}{e}$

Hence $\lim_{x \to 0} (\cot x)^{\frac{1}{\log x}} = \frac{1}{e}$

Example 2.43. *Evaluate* $\lim_{x \to \infty} \left(1 - \frac{1}{2x}\right)^{x+1}$

Solution. Let $\qquad f(x) = \lim_{x \to \infty} \left(1 - \frac{1}{2x}\right)^{x+1}, x > 1$

$\therefore \qquad \log f(x) = \lim_{x \to \infty} (x+1) \log\left(1 - \frac{1}{2x}\right), x > 1$

$$\log f(x) = \lim_{x \to \infty} \frac{\log\left(1 - \frac{1}{2x}\right)}{\frac{1}{x+1}} \qquad \left[\frac{0}{0} \text{ form}\right]$$

Apply L'Hospital Rule

$$= \lim_{x \to \infty} \frac{\frac{1}{\left(1 - \frac{1}{2x}\right)} \cdot \frac{1}{2x^2}}{-\frac{1}{(x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{2x}{(2x-1)} \cdot \frac{1}{2x^2}}{-\frac{1}{(x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{(x+1)^2}{-x(2x-1)} \quad \Rightarrow \quad \lim_{x \to \infty} \frac{x^2+1+2x}{-2x^2+x}$$

$$= \lim_{x \to \infty} \frac{1+\frac{1}{x^2}+\frac{2}{x}}{-2+\frac{1}{x}} \qquad \text{[Dividing numerator and denominator by } x^2\text{]}$$

$$= -\frac{1}{2}$$

Thus, $\log f(x) = -\frac{1}{2}$

$\Rightarrow$ $f(x) = e^{-1/2}$

$\Rightarrow$ $\lim_{x \to \infty} \left(1 - \frac{1}{2x}\right)^{x+1} = e^{-1/2}$

Example 2.44. *Evaluate* $\lim_{x \to \pi/2} (\sec x)^{\cot x}$.

Solution. Let $y = \lim_{x \to \pi/2} (\sec x)^{\cot x}$ $[\infty^0 \text{ form}]$

$\therefore$ $\log y = \lim_{x \to \pi/2} \cot x \log \sec x$ $[0 \times \infty \text{ form}]$

$$= \lim_{x \to \pi/2} \frac{\log \sec x}{\tan x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

Applying L'Hospital Rule,

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sec x} . \sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \sin x \cos x$$

$$= 0$$

Thus, $\log y = 0$

$\Rightarrow$ $y = e^0 = 1$

Hence $\lim_{x \to \pi/2} (\sec x)^{\cot x} = 1$

Example 2.45. *Prove that* $\lim_{x \to \infty} \left(\frac{2x+3}{2x+1}\right)^{3x+2} = e^3$

Solution. Let $y = \lim_{x \to \infty} \left(\frac{2x+3}{2x+1}\right)^{3x+2}$ $[1^\infty \text{ form}]$

$\therefore$ $\log y = \lim_{x \to \infty} (3x+2) \log\left(\frac{2x+3}{2x+1}\right)$

$$= \lim_{x\to\infty} (3x+2)\log\left(1+\frac{2}{2x+1}\right) \qquad [0\times\infty \text{ form}]$$

$$= \lim_{x\to\infty} (3x+2)\left[\frac{2}{2x+1} - \left(\frac{2}{2x+1}\right)^2 \cdot \frac{1}{2} + \ldots\right]$$

$$= \lim_{x\to\infty} \left[\frac{2(3x+2)}{(2x+1)} - \left(\frac{2}{2x+1}\right)^2 \cdot \frac{(3x+2)}{2} + \ldots\right]$$

$$= \lim_{x\to\infty} \left[\frac{2}{2x}\,\frac{\left(1+\frac{2}{3x}\right).3x}{\left(1+\frac{1}{2x}\right)} - \frac{2\,.\,3x\left(1+\frac{2}{3x}\right)}{(2x)^2\left(1+\frac{1}{2x}\right)^2} + \ldots\right]$$

$$= \lim_{x\to\infty} \left[\frac{3\left(1+\frac{2}{3x}\right)}{\left(1+\frac{1}{2x}\right)} - \frac{3}{2x}\,\frac{\left(1+\frac{2}{3x}\right)}{\left(1+\frac{1}{2x}\right)^2} + \ldots\right]$$

$$= 3$$

Thus, $\log y = 3$

$\Rightarrow$ $y = e^3$

Hence $\lim_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^{3x+2} = e^3$

Example 2.46. *Show that* $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = \frac{-e}{2}$.

Solution. Let $y = (1+x)^{1/x}$

$\therefore$ $\log y = \frac{1}{x}\log(1+x)$

$$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)$$

$$= 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$$

$\Rightarrow$ $y = e^{1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots}$

$= e\,.\,e^t$, where $t = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$

$$= e\left[1 + t + \frac{t^2}{2} + \ldots\right]$$

$$= e\left[1+\left(\frac{-x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)^2+\ldots\ldots\right]$$

$$= e\left[1-\frac{x}{2}+\frac{11}{24}x^2+\ldots\right]$$

$$= e-\frac{ex}{2}+\frac{11}{24}ex^2+\ldots\ldots$$

Now, $$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}\frac{\left(e-\frac{ex}{2}+\frac{11}{24}ex^2+\ldots\right)-e}{x}$$

$$= \lim_{x\to 0} -\frac{e}{2}+\frac{11}{24}ex+\ldots\ldots$$

$$= \frac{-e}{2}$$

Example 2.47. *Evaluate* $\lim\limits_{n\to\infty}\left[n+n^2 \log\frac{n}{n+1}\right]$.

Solution. Given, $\lim\limits_{n\to\infty}\left[n+n^2\log\frac{n}{n+1}\right]$

$$= \lim_{n\to\infty}\left[n+n^2\log\left(1-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[n+n^2\left(-\frac{1}{n+1}-\frac{1}{2(n+1)^2}-\frac{1}{3(n+1)^3}\ldots\right)\right]$$

$$= \lim_{n\to\infty}\left[n-\frac{n^2}{n+1}-\frac{n^2}{2(n+1)^2}-\frac{n^2}{3(n+1)^3}\ldots\right]$$

$$= \lim_{n\to\infty}\left[\frac{n}{n+1}-\frac{1}{2}\left(\frac{n}{n+1}\right)^2-\frac{1}{3}\frac{n^2}{(n+1)^3}\ldots\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}-\frac{1}{2}\left(\frac{1}{1+\frac{1}{n}}\right)^2-\frac{1}{3}\frac{1}{n\left(1+\frac{1}{n}\right)^3}\ldots\right]$$

$$= 1-\frac{1}{2}=\frac{1}{2}$$

Hence $$\lim_{n\to\infty}\left[n+n^2\log\frac{n}{n+1}\right] = \frac{1}{2}.$$

EXERCISE 2.7

1. Evaluate the following limits:

a. $\lim\limits_{x \to \pi/2} (\cos x)^{\cos x}$ b. $\lim\limits_{x \to 1} (x-1)^{x-1}$ c. $\lim\limits_{x \to \infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$

2. Determine the following limits:

a. $\lim\limits_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x}$ b. $\lim\limits_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$ c. $\lim\limits_{x \to \pi/2} (\sin x)^{\tan x}$ d. $\lim\limits_{x \to 0} \left(\frac{\sin hx}{x}\right)^{1/x}$

e. $\lim\limits_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$ f. $\lim\limits_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$ g. $\lim\limits_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$ h. $\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

i. $\lim\limits_{x \to 0} (1 + \sin x)^{\cot x}$ j. $\lim\limits_{x \to \infty} \left(1 + \frac{2}{x}\right)^{x}$ k. $\lim\limits_{x \to \pi/2} (\sin x)^{\tan^2 x}$

3. a. Show that $\lim\limits_{x \to 0^+} \dfrac{(1+x)^{1/x} - e + \dfrac{ex}{2}}{x^2} = \dfrac{11e}{24}$

b. Show that $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} - e + \dfrac{ex}{2} - \dfrac{11}{24}ex^2}{x^3} = \dfrac{-7}{16}e$

4. a. Prove that $\lim\limits_{x \to \infty} (1+x)^{1/x} = 1$ b. Prove that $\lim\limits_{x \to \infty} \left(1 + \frac{k}{x}\right)^{x} = e^k$

c. Prove that $\lim\limits_{x \to \infty} (x + e^x)^{2/x} = e^2$ d. Prove that $\lim\limits_{x \to \infty} \left(\frac{3x+1}{3x+4}\right)^{3x+2} = e^{-3}$

5. Evaluate:

a. $\lim\limits_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$ and then deduce $\lim\limits_{x \to 0} \left(\frac{2^x + 3^x}{2}\right)^{1/x} = \sqrt{6}$

b. $\lim\limits_{x \to \infty} \left(\frac{\log x}{x}\right)^{1/x}$ c. $\lim\limits_{n \to \infty} \dfrac{\log(1+n^2)}{n}$

Answers

1. a. 1 b. 1 c. 1

2. a. e^a b. 1 c. 1 d. 1

e. $1/e$ f. $e^{1/3}$ g. $e^{2/\pi}$ h. $e^{-1/6}$

i. e j. e^2 k. $e^{-1/2}$

5. b. 1 c. 0

INTERESTING FACTS

- Indeterminate forms are also found in physics. We can see its usage in quantum physics, particle decay, quantum mechanics, thermodynamics etc.
- Johann Bernoulli was also engaged in the creation of this unique rule.
- L'Hospital's Rule occasionally fails by falling into a never-ending cycle.
- Although written as Hospital, but the word is pronounced as "***Hopital***".

VIDEO REFERENCES

Indeterminate Forms Part I

APPLICATIONS TO REAL LIFE

- It has a significant application in commerce domain, where continuous compounding interest rates are encountered every day especially in investments, different types of bank accounts, while paying credit cards bills, mortgages, etc.
- It is used in Gamma functions which are further used in engineering, quantum physics, statistics, astrophysics, fluid dynamics, combinatory, probability theory, etc.

2.4 MAXIMA AND MINIMA

A function f is said to have a maximum value at $x = a$ if $f(a) > f(x)$ *i.e.*, $f(x) - f(a) < 0$ for all x in a small neighbourhood of a.

A function f is said to have a minimum at $x = a$ if $f(a) < f(x)$ *i.e.*, $f(x) - f(a) > 0$ for all values of x in a small neighbourhood of a.

In the adjoining fig., $f(x)$ has a maximum value at $x = a$ since $f(a)$ is greater than the neighbouring value of $f(x)$. Similarly $f(x)$ has a minimum at $x = b$ and maximum at d.

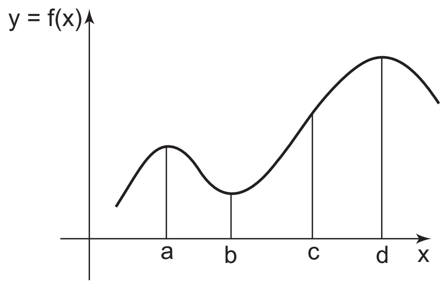


Fig. 2.7

Note that $f(x)$ has a maximum at $x = a$ even though $f(a) < f(c)$. The reason is that $f(a) > f(x)$ in a neighbourhood of a.

Thus a maximum value of $f(x)$ is not necessarily the greatest value of $f(x)$.

2.4.1 Condition for Maxima and Minima

Expanding by Taylor's theorem, about a point 'a', we have

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{1}{2!}\,(x-a)^2 f''(a) + \ldots$$

i.e.,
$$f(x) - f(a) = (x-a)\,f'(a) + \frac{1}{2!}\,(x-a)^2 f''(a) + \ldots$$

or $$= (x-a)\left\{f'(a)+\frac{1}{2!}(x-a)f''(a)+\ldots\right\} \quad \ldots(1)$$

When $x - a$ is small, $f'(a)$ is numerically more than the succeeding terms. So the sign of $f(x) - f(a)$ depends upon $(x - a) f'(a)$. But this will have only one sign when $x > a$ and another when $x < a$. Therefore no maximum or minimum is possible at $x = a$ unless $f'(a) = 0$.

If $f'(a) = 0$, then (1) becomes

$$f(x)-f(a) = (x-a)^2\left\{\frac{1}{2}f''(a)+\frac{1}{6}(x-a)f'''(x)+\ldots\right\} \quad \ldots(2)$$

For small values of $x - a$, $\frac{1}{2} f''(a)$ has more numerical value than the succeeding terms. So the sign of $f(x) - f(a)$ depends on $\frac{1}{2} (x - a)^2 f''(a)$ or $f''(a)$, as $\frac{1}{2} (x - a)^2$ is always positive.

Hence the function $f(x)$ has maxima, minima at $x = a$ if

i. $f'(a) = 0$ and $f''(a) =$ negative, $f(x)$ has maximum at 'a'.
ii. $f'(a) = 0$ and $f''(a) =$ positive, $f(x)$ has minimum at a.
iii. $f'(a) = 0$ and $f''(a) = 0$, $f(x)$ has neither maximum nor minimum at $x = a$ unless $f'''(a) = 0$. The sign of $f^{iv}(a)$ will then determine the nature of $f(x)$.

2.4.2 First Derivative Test for Extrema (Maxima or Minima)

Let f be a continuous function defined on Interval $I = (a, b)$ and let c be the critical point in I *i.e.* $c \in I$, then

i. $f'(x)$ changes sign from positive to negative as x increases through c *i.e.*, $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$, then c is the point of local maxima and f has local maximum at c.
ii. $f'(x)$ changes sign from negative to positive as x increases through c *i.e.*, $f'(x) \leq 0$ for $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c + \delta)$, then c is the point of local minima and f has local minimum at c.
iii. $f'(x)$ does not change sign as x increases through c, then f has no extremum (neither maximum nor minimum) at c and such point is called point of inflection.

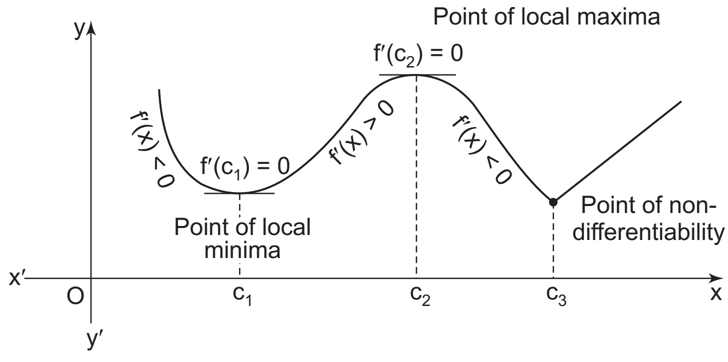


Fig. 2.8

Definition. Critical point of f: Let f be a continuous function defined on an interval I and $c \in I$, then c is called critical point if either $f'(c) = 0$ or f is not differentiable at points in I.

Definition. Let 'f' be real valued function and let c be an interior point of domain of f, then

a. c is called a point of local maxima, if there is an $h > 0$, such that $f(c) > f(x)$, $\forall\, x$ in $(c - h, c + h)$. The value $f(c)$ is called the local maximum value of f.

b. c is called a point of local minima if there is an $h > 0$ such that $f(c) < f(x)$, $\forall\, x$ in $(c - h, c + h)$.

The value $f(c)$ is called the local minimum value of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of 'f' around c will be as shown in given figure.

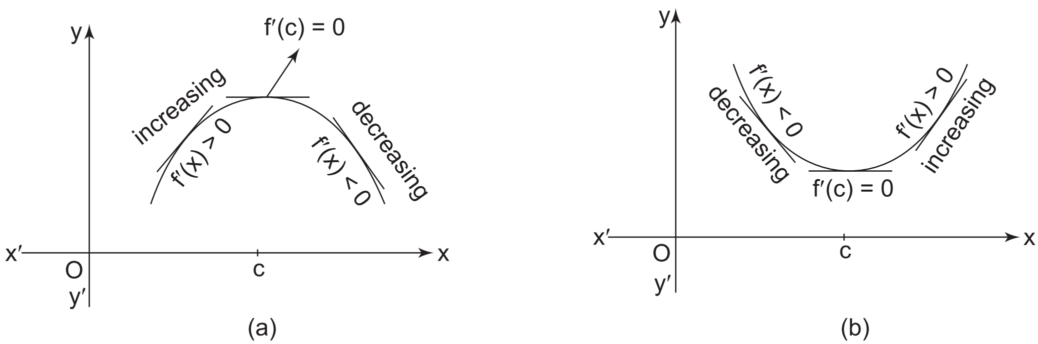


Fig. 2.9

Note that the function f is increasing (*i.e.*, $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (*i.e.*, $f'(x) < 0$) in the interval $(c, c + h)$. It suggests that $f'(c)$ must be zero.

Example 2.48. *Find all points of local maxima and minima of the function, when f is given by*

$$f(x) = x^3 - 3x + 3.$$

Solution. Given $\quad f(x) = x^3 - 3x + 3$

Differentiating w.r.t. x, we get

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x - 1)(x + 1) \end{aligned}$$

Now $\quad f'(x) = 0 \quad \Rightarrow \quad 3(x - 1)(x + 1) = 0$

$\Rightarrow \quad x = 1, -1$

Thus, $x = \pm 1$ are only critical points and function can have maximum or minimum values at $x = 1, -1$.

Applying 1[st] derivative test

Value of x		Sign of $f'(x) = 3(x - 1)(x + 1)$
$x = 1$	→ to left (say 0.98)	$f'(x) < 0$
	→ to right (say 1.01)	$f'(x) > 0$
$x = -1$	→ to left (say –1.01)	$f'(x) > 0$
	→ to right (say –0.9)	$f'(x) < 0$

At $x = 1, f'(x)$ changes sign from negative to positive.

$\therefore$ $x = 1$ is a point of local minima, and $f(1) = (1)^3 - 3(1) + 3 = 1$ is the local minimum value.

At $x = -1, f'(x)$ changes sign from positive to negative.

$\therefore$ $x = -1$ is the point of local maxima and $f(-1) = (-1)^3 - 3(-1) + 3 = 5$ is the local maximum value.

Example 2.49. *Find all the points of local maxima and local minima of the function f, which is given by $f(x) = x^3 + 1$.*

Solution. Given $f(x) = x^3 + 1$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2$$

Now, $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$

Thus, $x = 0$ is the only critical point of 'f' and function can have maximum or minimum value at $x = 0$.

On applying 1st derivative test,

Value of x	Sign of $f'(x) = 3x^2$
$x = 0$ → to left (say –0.1)	> 0
$x = 0$ → to right (say 0.1)	> 0

Thus, at $x = 0, f'(x)$ does not changes its sign. So $x = 0$ is neither a point of local maxima nor a point of local minima.

Thus, $x = 0$ is a point of inflection.

Remark:

i. Consider a function $f(x) = x^2, x \in \mathbb{R}$

Clearly, f has minimum value at $x = 0$ and $f(0) = 0$ but f has no maximum value in $\mathbb{R}$.

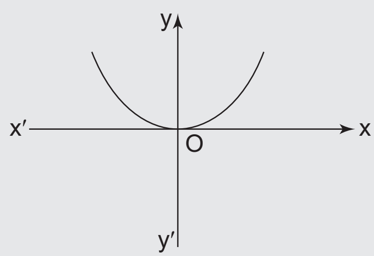


Fig. 2.10

ii. Consider a function $f(x) = |x|, x \in \mathbb{R}$

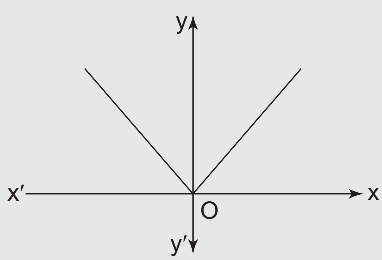


Fig. 2.11

f has a minimum value at $x = 0$. Also $f(0) = |0| = 0$, but f has no maximum value in R.

iii. Consider a function $f(x) = x, x \in (0, 2)$

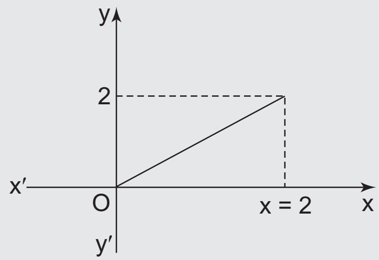


Fig. 2.12

f has neither maximum value nor minimum value in (0, 2)

$$f(x) = x, x \in (0, 2)\ f'(x) = 1.$$

At all points in (0, 2), $f'(x) > 0$ *i.e.*, no changes in sign.

$\therefore$ f does not have maxima or minima in (0, 2).

2.4.3 Second Derivative Test for Extrema (Maxima or Minima)

Let f be a twice differentiable function defined on Interval I and $c \in I$. Then

i. if $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima and $f(c)$ is local maximum value of f.

ii. if $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima and $f(c)$ is local minimum value of f.

iii. if $f'(c) = 0$ and $f''(c) = 0$, then test fails. Further process need to be done to check for extremum values.

Example 2.50. *Find local maximum and local minimum value of the function f, given by*

$$f(x) = x^5 - 5x^4 + 5x^3 + 10, x \in \mathbb{R}.$$

Solution. Given $f(x) = x^5 - 5x^4 + 5x^3 + 10$

Now, $f'(x) = 5x^4 - 20x^3 + 15x^2$

$= 5x^2 (x - 3) (x - 1)$

$\therefore$ $f'(x) = 0 \Rightarrow 5x^2 (x - 3) (x - 1) = 0$

$\Rightarrow$ $x = 0, 1, 3$

Now $f''(x) = 20x^3 - 60x^2 + 30x$

$$= \begin{cases} f''(0) = 0 \\ f''(1) = -10 < 0 \\ f''(3) = 90 > 0 \end{cases}$$

By second derivative test, $x = 1$, is a point of local maxima and maximum value is

$$f(1) = 11.$$

and $x = 3$ is a point of local minima and minimum value is

$$f(3) = -17$$

At $x = 0$, second derivative test fails.

Now, by first derivative test

Value of x	Sign of $f'(x) = 5x^2(x-3)(x-1)$
$x = 0$ → left (say –0.1)	> 0
$x = 0$ → right (say 0.1)	> 0

Thus at $x = 0$, f has neither maxima nor minima.

Example 2.51. *Find local maximum or local minimum value of function f given by* $f(x) = |x + 2| - 1$.

Solution. Here, $\quad f(x) = |x + 2| - 1$

or $\quad f(x) = \begin{cases} x+2-1 & x \geq -2 \\ -(x+2)-1 & x < -2 \end{cases}$

or $\quad f(x) = \begin{cases} x+1 & x \geq -2 \\ -x-3 & x < -2 \end{cases}$

Now, $\quad f'(x) = \begin{cases} 1 & x > -2 \\ -1 & x < -2 \end{cases}$

By first derivative test,

Value of x	Sign of $f'(x)$
$x = -2$ → left (say –2.1)	< 0
$x = -2$ → right (say –1.9)	> 0

$f'(x)$ changes sign from negative to positive.

$\therefore \quad f(x)$ has local minima at $x = -2$ and local minimum value is

$$f(-2) = |-2 + 2| - 1 = -1$$

$f(x)$ has no local maxima and hence no maximum value.

We can also check in the following way:

Here $\quad f(x) = |x + 2| - 1$

We know that $\quad |x + 2| \geq 0$

For minimum, $\quad x + 2 = 0$

$\Rightarrow \quad x = -2$

Thus, f has minimum value at –2, and $f(-2) = -1$.

Example 2.52. *Find maximum and minimum value of function* $f(x) = \sin 2x + 5$.

Solution. Here, $\quad f(x) = \sin 2x + 5$

We know that $\quad -1 \leq \sin x \leq 1$

$\therefore \quad -1 \leq \sin 2x \leq 1$

On adding (5) on both sides,

$$-1 + 5 < \sin 2x + 5 \leq 1 + 5$$

$$4 \leq \sin 2x + 5 \leq 6$$

Thus, $f(x)$ has maximum value 6 and minimum value 4.

Example 2.53. *Find maximum and minimum value of the function* $f(x) = x\sqrt{1-x}$, $x > 0$.

Solution. Here

$$f(x) = x\sqrt{1-x}, x > 0$$

$$f(x) = \sqrt{1-x} + \frac{x(-1)}{2\sqrt{1-x}}$$

$$= \sqrt{1-x} + \frac{(-x)}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f''(x) = 0 \Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0$$

$$\Rightarrow \quad 2-3x = 0 \Rightarrow x = 2/3$$

Now,

$$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}\,(-3) - (2-3x)\cdot\dfrac{-1}{2\sqrt{1-x}}}{(1-x)}\right]$$

$$= \frac{1}{2}\left[\frac{-6\,(1-x)+(2-3x)}{2(1-x)^{3/2}}\right]$$

$$= \frac{1}{2}\left[\frac{-4+3x}{2(1-x)^{3/2}}\right]$$

At $x = 2/3$,

$$f''(2/3) = \frac{1}{2}\left(\frac{-2}{2\left(\frac{1}{3}\right)^{3/2}}\right) > 0$$

By second derivative test, $x = 2/3$ is a point of maxima and f has maximum values at $x = 2/3$

$$\therefore \quad f(2/3) = \frac{2}{3}\sqrt{1-2/3} = \frac{2}{3}\sqrt{\frac{1}{3}}$$

$$= \frac{2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \qquad \text{[Rationalisation]}$$

$$= \frac{2\sqrt{3}}{9}.$$

EXERCISE 2.8

1. Find maximum value of $(x-1)(x-2)(x-3)$.
2. Show that $x^3 - 3x^2 + 3x + 7$ has neither a maximum nor minimum at $x = 1$.
3. Show that $\sin x\,(1 + \cos x)$ has maximum at $x = \frac{1}{3}\pi$.

4. Find maximum and minimum value of following functions:

i. $f(x) = (2x - 1)^2 + 3$

ii. $f(x) = 9x^2 + 12x + 2$

iii. $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

5. Find all the points of local maxima and minima and also find the maximum and minimum values of the following functions:

i. $f(x) = x^3 - 3x$

ii. $f(x) = \sin x - \cos x, x \in (0, 2\pi)$

iii. $f(x) = \dfrac{x}{2} + \dfrac{2}{x}, x > 0$

iv. $f(x) = \dfrac{1}{x^2 + 2}$

v. $f(x) = \dfrac{x}{(1 + x^2)^2}$

vi. $f(x) = (x - 1)^4 (x - 2)^2$

vii. $f(x) = 2x^3 - 6x^2 + 6x + 5$

6. Examine whether 'f' has local maximum or minimum at 0

i. $f(x) = \begin{cases} 2x + 3, & x > 0 \\ -3x + 1, & x \leq 0 \end{cases}$

ii. $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ -3x + 1, & x > 0 \end{cases}$

iii. $f(x) = \begin{cases} 2x + 3, & x \geq 0 \\ -3x + 1, & x < 0 \end{cases}$

iv. $f(x) = |x| + |x - 1|$

7. Find the maximum/minimum values of the following functions:

i. $f(x) = -|x + 1| + 3$

ii. $f(x) = |\sin 4x + 3|$

iii. $f(x) = x + 1, x \in (-1, 1)$

Answers

1. $\dfrac{2}{3\sqrt{3}}$

4. i. min. value = –2

ii. min. value = –2

iii. min. value = 1/3, max. value = 3

5. i. min at 1, $f(1) = -2$, max. at –1, $f(-1) = 2$

ii. min. at $\dfrac{7}{4}\pi, f\left(\dfrac{7}{4}\pi\right) = -\sqrt{2}$, max. at $\dfrac{3}{4}\pi, f\left(\dfrac{3}{4}\pi\right) = \sqrt{2}$

iii. min. at $x = 2, f(2) = 1$

iv. max. at 0, $f(0) = \dfrac{1}{2}$

v. max. at 1, $f(1) = \dfrac{1}{4}$, min. at –1, $f(-1) = -\dfrac{1}{4}$

vi. max. at 1, $\dfrac{5}{3}$, $f(1) = 0, f\left(\dfrac{5}{3}\right) = \dfrac{16}{729}$, min. at $x = 2, f(2) = 0$

vii. neither maxima nor minima

6. i. min.

ii. max.

iii. neither max. nor min.

iv. minimum

7. i. max. value = 3, no min. value ii. min. value = 2, max. value = 4
iii. neither max. nor min. value

SOME MISCELLANEOUS EXAMPLES

Based on Maxima-Minima

Example 2.54. *Show that of all the rectangles with a given perimeter, the square has the largest area.*

Solution. Try yourself.

Example 2.55. *Show that of all the rectangles of a given area, the square has the smallest perimeter.*

Solution. Try yourself.

Example 2.56. *If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is (π/3)*

Solution. Try yourself.

Example 2.57. *Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.*

Solution. Let $ABCD$ be a rectangle inscribed in a given circle with centre O and radius r.

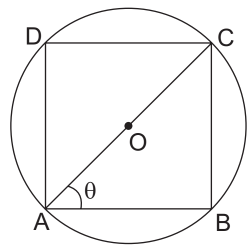


Fig. 2.13

Let $\angle CAB = \theta$.

Then, $AC = 2r$, $AB = 2r \cos\theta$ and $BC = 2r \sin\theta$.

Let A be the area of rectangle $ABCD$.

Then, $A = AB \times BC = 4r^2 \sin\theta \cos\theta = 2r^2 \sin 2\theta$.

Thus, $A = 2r^2 \sin 2\theta$, where r is constant.

$$\therefore \quad \frac{dA}{d\theta} = 4r^2 \cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -8r^2 \sin 2\theta.$$

Now, $$\frac{dA}{d\theta} = 0 \quad \Rightarrow \quad 4r^2 \cos 2\theta = 0$$

$$\Rightarrow \quad \cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}, \text{ i.e., } \theta = \frac{\pi}{4}.$$

And, $$\left[\frac{d^2A}{d\theta^2}\right]_{\theta = (\pi/4)} = -8r^2 \sin\frac{\pi}{2} = -8r^2 < 0.$$

$\therefore$ $\theta = (\pi/4)$ is a point of maximum.

Thus, area is maximum when $\theta = (\pi/4)$.

In this case, $$AB = 2r \cos\frac{\pi}{4} = r\sqrt{2}$$

and, $$BC = 2r \sin\frac{\pi}{4} = r\sqrt{2}$$

Thus, $AB = BC$ and therefore, $ABCD$ is a square.

Example 2.58. *Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.*

Solution. Let ABC be a triangle inscribed in a given circle with centre O and radius r.

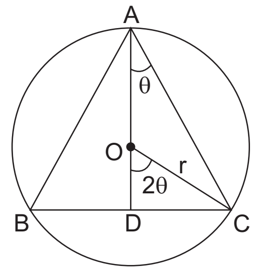


Fig. 2.14

For maximum area, the vertex A should be at a maximum distance from the base BC.

Therefore, A must lie on the diameter, perpendicular to BC. Thus, $AD \perp BC$.

So, triangle ABC must be isosceles.

Let $\quad \angle CAD = \theta.$

Now, $\quad BC = 2CD = 2OC \sin 2\theta = 2r \sin 2\theta$

and, $\quad AD = (OA + OD) = (r + r\cos 2\theta).$

Let A be the area of the triangle.

Then, $$A = \frac{1}{2}\, BC \times AD = r^2 \sin 2\theta\,(1 + \cos 2\theta).$$

$$\therefore \quad \frac{dA}{d\theta} = r^2\,[\sin 2\theta\,(-2\sin 2\theta) + (1 + \cos 2\theta)\,.\,2\cos 2\theta]$$
$$= r^2\,[2\,(\cos^2 2\theta - \sin^2 2\theta) + 2\cos 2\theta] = 2r^2\,[\cos 4\theta + \cos 2\theta]$$

And, $$\frac{d^2A}{d\theta^2} = 2r^2\,[-4\sin 4\theta - 2\sin 2\theta] = -4r^2\,(2\sin 4\theta + \sin 2\theta)$$

Now, $$\frac{dA}{d\theta} = 0 \quad \Rightarrow \quad \cos 4\theta + \cos 2\theta = 0$$

$$\Rightarrow \quad \cos 4\theta = -\cos 2\theta = \cos(\pi - 2\theta)$$

$$\Rightarrow \quad 4\theta = \pi - 2\theta \quad \Rightarrow \quad \theta = \frac{\pi}{6}.$$

and $$\left[\frac{d^2A}{d\theta^2}\right]_{\theta = (\pi/6)} = -4r^2\left(2\sin\frac{2\pi}{3} + \sin\frac{\pi}{3}\right) = -6r^2\sqrt{3} < 0$$

$$\therefore \quad \theta = \frac{\pi}{6} \text{ is a point of maximum.}$$

So, in this case, each angle of the triangle is $(\pi/3)$.

Hence, ABC is an equilateral triangle.

Based on Volume and Area of Solids

Example 2.59. *Show that a cylinder of a given volume which is open at the top has minimum total surface area, provided its height is equal to the radius of its base.*

Solution. Try yourself.

Example 2.60. *Show that the height of a cylinder which is open at the top, having a given surface and the greatest volume, is equal to the radius of its base.*

Solution. Try yourself.

Example 2.61. *Show that the height of a cylinder which is open at the top, having a given surface area and maximum volume is* $\sin^{-1}(1/3)$.

Solution. Try yourself.

Example 2.62. *Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.*

Solution. Let V be the fixed volume of a closed cuboid with length a, breadth a and height h.

Let S be its surface area.

Then, $$V = (a \times a \times h) \text{ or } h = \frac{V}{a^2} \quad \text{...(1)}$$

Now, $$S = 2(a^2 + ah + ah) = 2(a^2 + 2ah) = 2\left(a^2 + \frac{2V}{a}\right) \quad \text{[using (1)]}$$

i.e., $$S = 2\left(a^2 + \frac{2V}{a}\right). \therefore \frac{dS}{da} = 2\left(2a - \frac{2V}{a^2}\right) \text{ and } \frac{d^2S}{da^2} = \left(4 + \frac{8V}{a^3}\right).$$

Now, $$\frac{dS}{da} = 0 \Rightarrow V = a^3 \Rightarrow a \times a \times h = a^3 \Rightarrow h = a.$$

Now, when $h = a$, we have

$$V = a^3$$

$$\therefore \left[\frac{d^2S}{da^2}\right]_{h=a} = \left(4 + \frac{8a^3}{a^3}\right) = 12 > 0.$$

So, S is minimum when length = a, breadth = a and height = a, i.e., when it is a cube.

Example 2.63. *Show that the height of a closed cylinder of given surface and maximum volume is equal to the diameter of its base.*

Solution. Try yourself.

Example 2.64. *Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere. Find the volume of the largest cone inscribed in a sphere of radius R.*

Solution. Let R be the radius of the given sphere with centre O, and let V be the volume of the inscribed cone, h be its height and r be the radius of the base.

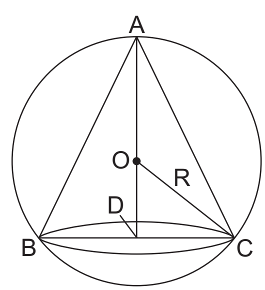


Fig. 2.15

In the given figure, we have

$$OD = AD - AO = (h - R)$$

$$\therefore R^2 = (h - R)^2 + r^2 \text{ or } r^2 = h(2R - h) \quad \text{...}(i)$$

Now, $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2 (2R - h) \quad \text{[using (1)]}$$

$$\therefore \frac{dV}{dh} = \frac{1}{3}\pi h (4R - 3h),$$

and $$\frac{d^2V}{dh^2} = \left(\frac{4}{3}\pi R - 2\pi h\right).$$

For a maxima or minima, we have

$$\frac{dV}{dh} = 0$$

Now, $$\frac{dV}{dh} = 0 \quad \Rightarrow \quad \frac{1}{3}\pi h(4R - 3h) = 0$$

$$\Rightarrow \quad h = 0 \text{ or } (4R - 3h) = 0 \quad \Rightarrow \quad h = \frac{4}{3}R \qquad [\because \quad h \neq 0]$$

and $$\left[\frac{d^2V}{dh^2}\right]_{h=(4/3)R} = -\frac{4\pi R}{3} < 0.$$

So, V is maximum when $h = \frac{4}{3}R$, *i.e.*, when $3h = 2(2R)$

i.e., 3 times the height = 2 times the diameter.

Volume of the largest cone $= \frac{1}{3}\pi \times \frac{16R^2}{9} \times \left(2R - \frac{4R}{3}\right) = \frac{32\pi R^3}{81}$

EXERCISE 2.9

1. Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.
2. Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.
3. Prove that the largest rectangle with a given perimeter is a square.
4. Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.
5. Prove that the perimeter of a right-angled triangle of given hypotenuse is a maximum when the triangle is isosceles.
6. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base
7. Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, -8)$.
8. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

Answers

1. 6, 2 **2.** $\frac{ap}{p+q}, \frac{aq}{p+q}$ **7.** $(4, -4)$

INTERESTING FACTS

- Maximum and minimum value that are found using Maxima and Minima are together known as Extrema (plural of extremum).
- A sculpture was displayed in the World Expo 2017 in Astana, Kazakhstan, which was named as Minima | Maxima, whose design was unique in itself.
- (https://architizer.com/projects/minima-maxima-1/)

VIDEO REFERENCES

Extreme values-I

APPLICATIONS TO REAL LIFE

- Applications in medicine while studying the effectiveness of drugs /spread of diseases (*i.e.*, after how much time the maximum efficiency has been observed).
- Decay study in nuclear energy sector.
- In business, industry uses this concept for maximising their profits or minimizing their loss, by estimating prices for items and also how many to keep in stock.
- Population Growth Curve.

SUBJECTIVE SOLVED QUESTIONS (HOTS)

Example 1. *Let $f : [2, 5] \to R$ be continuous and differentiable on (2, 5). Assume that $f'(x) = (f(x))^2 + \pi$ for all $x \in (2, 5)$. Find $f(5) - f(2)$.*

Solution. Given that f is continuous on [2, 5] and differentiable on (2, 5).

Hence, by Lagrange's mean value theorem, there exists atleast one $c \in (2, 5)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c). \text{ Here } b = 5, a = 2$$

then $$\frac{f(5)-f(2)}{5-2} = (f(c))^2 + \pi$$

$$\frac{f(5)-f(2)}{3} \geq \pi \qquad [\because (f(c))^2 \geq 0 \ \forall \ c \in (2, 5)]$$

$$\therefore \quad f(5) - f(2) \geq 3\pi.$$

Example 2. *Prove that between any two roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x - 1 = 0$.*

Solution. Given that

$$e^x \cos x = 1$$

After rearranging, we have

$$\cos x = e^{-x}$$

Consider $$f(x) = \cos x - e^{-x}$$

i. cosine function and exponential function both are continuous and differentiable on $\mathbb{R}$, so on every interval say $[a, b]$.

ii. assuming that a, b are root of f.

Then $$f(a) = f(b) = 0$$

Hence by Rolle's theorem, $\exists\, c \in (a, b)$, such that $f'(c) = 0$

Now, $$f'(x) = -\sin x + e^{-x}$$

$\Rightarrow$ $$-\sin c + e^{-c} = 0$$

$\Rightarrow$ $$e^{-c} = \sin c$$

or $$1 = e^c \sin c$$

i.e., $$e^c \sin c - 1 = 0$$

This imply that $e^x \sin x = 1$ has one root $c \in (a, b)$ *i.e.*, between two roots of $e^x \cos x = 1$.

Example 3. *Prove that the quadratic equation $3px^2 + 2qx + r = 0$ has a root in $(0, 1)$ if $p + q + r = 0$.*

Solution. Let $$f(x) = px^3 + qx^2 + rx$$

i. Clearly $f(x)$ is continuous in $[0, 1]$, being a polynomial in x.

ii. $$f'(x) = 3px^2 + 2qx + r$$

$f(x)$ is differentiable in $(0, 1)$, again, being a polynomial in x.

iii. $$f(0) = p(0)^3 + q(0)^2 + r(0) = 0$$

$$f(1) = p(1)^3 + q(1)^3 + r(1) = p + q + r = 0$$

Hence, Rolle's theorem satisfied, there exists atleast one $c \in (0, 1)$ such that $f'(c) = 0$

$$f'(c) = 3pc^2 + 2qc + r$$

as $$f'(c) = 0 \quad \Rightarrow \quad 3pc^2 + 2qc + r = 0$$

which is quadratic in c and $c \in (0, 1)$.

Thus, $3px^2 + 2qx + r = 0$ has root in $(0, 1)$.

Example 4. *Let the function f be continuous on the closed interval $[a, b]$, differentiable on open interval (a, b) and $f'(x) = 0$ for all $x \in (a, b)$. Then show that f is constant on $[a, b]$.*

Solution. To show that f is constant on $[a, b]$, it is sufficient to prove that $f(x) = f(a)\ \forall\, x \in [a, b]$

Let $x \in [a, b]$ such that $x > a$.

Now, applying mean value theorem on $[a, x]$ as

i. f is continuous on $[a, b]$

$\therefore$ f is continuous on $[a, x]$.

ii. It is given that f is differentiable on (a, b)

$\therefore$ f is differentiable on (a, x).

By Lagrange's mean value theorem, there exists atleast one $c \in (a, x)$ such that

$$\frac{f(x)-f(a)}{x-a} = f'(c) \qquad \text{...(1)}$$

Since, $\qquad f'(c) = 0 \qquad [\because \; f'(x) = 0 \; \forall \; x \in (a, b)]$

From (1), $\qquad \dfrac{f(x)-f(a)}{x-a} = 0$

$$f(x) - f(a) = 0$$

$\Rightarrow \qquad f(x) = f(a)$ for any $x \in [a, b]$.

$\therefore \quad f$ is constant on $[a, b]$.

Example 5. *Let* $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$ *and let* $g(x) = x^2$ *for* $x \in [0, 1]$. *Then both f and g are differentiable on [0, 1] and* $g(x) > 0$ *for* $x \neq 0$. *Show that* $\lim\limits_{x \to 0} f(x) = 0 = \lim\limits_{x \to 0} g(x)$ *and that* $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)}$ *does not exist.*

Solution. Here $\qquad f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \text{ for } x \neq 0$$

Also, $\qquad g(x) = x^2 \; \forall \; x \in [0, 1]$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 = 0 \; \forall \; x$$

Now, for $x \neq 0$,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x^2} = \lim_{x \to 0} \sin \frac{1}{x}$$

$= \sin (\infty)$

$=$ oscillates between -1 and 1

$=$ Does not exist.

Example 6. *Using the remainder of Maclaurin polynomial of* n^{th} *order for f(x) defined as*

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{n+1}(c), \; n \ge 0, \; 0 \le c \le x.$$

What is the order of the Maclaurin polynomial at least required to get an absolute true error of atmost 10^{-6} *in the calculation of sin (0.1).*

Solution. Given that $\; R_n(x) = \dfrac{x^{n+1}}{(n+1)!} f^{n+1}(c), n \ge 0, 0 \le c \le x$

$$R_n(0.1) = \frac{(0.1)^{n+1}}{(n+1)!} f^{n+1}(c), n \ge 0, 0 \le c \le 0.1$$

Since derivative of $f(x)$ are simply $\sin x$ and $\cos x$ and $|\sin x| \le 1$ and $|\cos x| \le 1$

$\therefore \qquad |f^{n+1}(c)| \le 1$

Now, $\qquad R_n(0.1) \le \dfrac{(0.1)^{n+1}}{(n+1)!}(1)$

$$= \frac{(0.1)^{n+1}}{(n+1)!}$$

So, when $\qquad R_n(0.1) < 10^{-6}$

i.e., $\qquad \dfrac{(0.1)^{n+1}}{(n+1)!} < 10^{-6}$

This is possible only when $n \ge 4$.

But since Maclaurin polynomial for sin x only include odd terms, therefore, $n \ge 5$.

Example 7. *Let $f: (0, \infty) \to R$ be differentiable. Suppose that $\lim\limits_{x \to \infty} (f(x) + f'(x)) = L$. Show that $\lim\limits_{x \to \infty} f(x) = L$ and $\lim\limits_{x \to \infty} f'(x) = 0$.*

Solution. Try yourself.

Example 8. *Let $f: [a, b] \to R$ be thrice differentiable function such that $f(a) = f(b) = 0$ and $f'(a) = f'(b) = 0$, then prove that there exist $c \in (a, b)$ such that $f'''(c) = 0$.*

Solution. Here, $f: [a, b] \to R$ be thrice differentiable function.

$\Rightarrow$ f is continuous on $[a, b]$ and f is differentiable on (a, b) and also $f(a) = f(b) = 0$ then by Rolle's theorem, there exist atleast one $c_1 \in (a, b)$ such that

$$f'(c_1) = 0$$

Again applying Rolle's theorem for f'

i. f' is continuous on $[a, b]$ and, therefore, continuous on $[a, c_1]$ and $[c_1, b]$.

ii. f' is differentiable on (a, b), therefore, differentiable on (a, c_1) and (c_1, b)

iii. $f'(a) = f'(c_1) = f(b) = 0$

By Rolle's theorem, there exists atleast one $c_2 \in (a, c_1)$ and one $c_3 \in (c_1, b)$ such that

$$f''(c_2) = 0 \text{ for } c_2 \in (a, c_1)$$

and $\qquad f''(c_3) = 0 \text{ for } c_3 \in (c_1, b)$

Similarly, applying Rolle's theorem on f'' on (c_2, c_3), f'' being continuous on $[c_2, c_3]$ and differentiable on (c_2, c_3) and $f''(c_2) = f''(c_3) = 0$ then there exists atleast one $c \in (c_2, c_3)$ such that

$$f'''(c) = 0 \text{ for } c \in (c_2, c_3)$$

Thus, $\qquad f'''(c) = 0 \text{ for } c \in (a, b)$

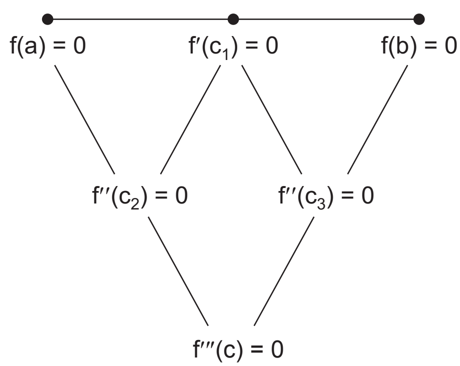


Fig. 2.16

Example 9. *A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also find this maximum volume.*

Solution. Let each side of the square to be cut off = x

$\therefore$ For the box, length = $24 - 2x$

breadth = $24 - 2x$, height = x

Let V be the volume of the box

$$\therefore \quad V = x(24 - 2x)^2$$

$$\therefore \quad \frac{dV}{dx} = x \,.\, 2(24 - 2x)(-2) + (24 - 2x)^2 \,.\, 1$$

$$= (24 - 2x)(24 - 6x)$$

For maxima or minima, $\dfrac{dV}{dx} = 0$

$\Rightarrow$ $(24 - 2x)(24 - 6x) = 0 \quad \Rightarrow \quad x = 4, 12$

$x = 4 \qquad [\because \quad x = 12$ cm is not possible$]$

Also, $$\frac{d^2V}{dx^2} = (24 - 2x)(-6) + (24 - 6x)(-2)$$

At $x = 4$ $$\frac{d^2V}{dx^2} = (24 - 8)(-6) = -\text{ve}$$

$\Rightarrow$ V is maximum when $x = 4$

$\therefore$ Volume is maximum when square of side 4 cm is cut from each corner.

$\therefore$ Maximum volume = $4(24 - 8)^2 = 1024$ cu. cm.

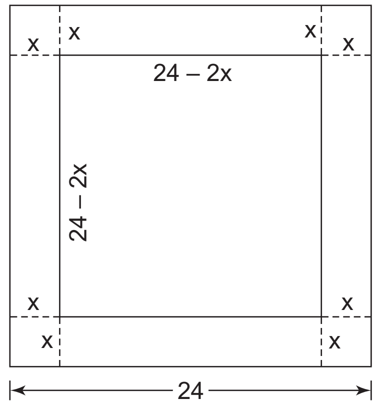


Fig. 2.17

Example 10. *A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs ₹ 70 per sq. m, for the base and ₹ 45 per sq. m. for sides, what is the cost of least expensive tank?*

Solution. Let x mt and y mt be the sides of the base of the tank.

The depth of the tank is given to be 2 m

Volume of tank = $2xy$ m^3

$\Rightarrow \quad 2xy = 8$

$\Rightarrow \quad xy = 4 \quad [\because$ Volume of tank = 8 m^3 (given)$]$

Area of base of tank = $(x \times y)$ m^2

Now, it is given that cost of building the base of tank is ₹ 70 per sq. m.

$\therefore$ Cost of building the base of tank = ₹ $70\,xy$

Total area of the sides of the tank

$= (2x + 2x + 2y + 2y)$ m^2 $= 4(x + y)$ m^2

It is given that the cost of building the sides of tank is ₹ 45 per sq. m.

$\therefore$ Cost of building the sides of the tank = ₹ $180(x + y)$

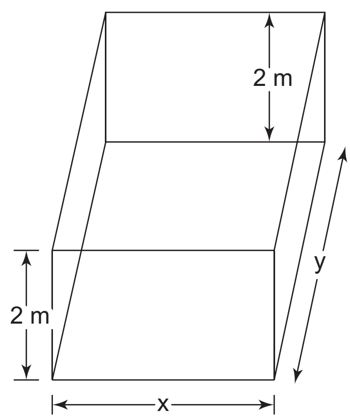


Fig. 2.18

Let C be the total cost of building the tank

$$\therefore \quad C = ₹\, [70xy + 180(x + y)]$$

$$= ₹ \left[70(4) + 180\left(x + \frac{4}{x}\right)\right] \qquad [\because \quad xy = 4]$$

$$= ₹\left[280+180x+\frac{720}{x}\right]$$

$\therefore$ $$\frac{dC}{dx} = 180 - \frac{720}{x^2}$$

For maxima or minima, $\frac{dC}{dx} = 0$

i.e., $180 - \frac{720}{x^2} = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$ [$\because$ x cannot be –ve]

Now, $$\frac{d^2C}{dx^2} = \frac{1440}{x^3}$$

At $x = 2$, $$\frac{d^2C}{dx^2} = \frac{1440}{(2)^3} = +\text{ve}$$

$\therefore$ C (*i.e.*, total cost) is minimum when $x = 2$

Hence, the cost of the least expensive tank = ₹ $\left[280+180(2)+\frac{720}{2}\right]$ = ₹ 1000.

SUMMARY

1. Necessary condition for applying the Rolle's theorem, mean value theorem, is that function should be continuous and differentiable.
2. Expansion of $f(x)$ in powers of x (Maclaurin's series)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

3. Expansion of function $f(x)$ about $x = a$ in powers of $(x - a)$ (Taylor's series)

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

4. Indeterminate forms

i. $\frac{0}{0}$ ii. $\frac{\infty}{\infty}$ iii. $0 \times \infty$ iv. $\infty - \infty$

v. 1^∞ vi. 0^0 vii. ∞^0

5. L'Hospital's Rule

If $\lim\limits_{x\to a} f(x) = f(a) = 0$ and $\lim\limits_{x\to a} g(x) = g(a) = 0$ then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a^+} \frac{f'(x)}{g'(x)}$

6. Standard Results on Limit

a. $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ b. $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$ c. $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \log a$ d. $\lim\limits_{x\to 0} \left(1+\frac{1}{x}\right)^x = e$

7. Maxima and Minima: For the function $f(x)$, we find $f'(x)$ and equate it to zero *i.e.*, $f'(x) = 0$ which gives the critical points of $f(x)$. Now on these critical points we check maxima and minima of $f(x)$.

Case I. First find $f''(x)$ on critical points,

If $f''(x) > 0$, then critical point is point of minima.

Case II. If $f''(x) < 0$, then critical point is point of maxima.

8. Some standard expansion of functions:

a. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

b. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

c. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

d. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

e. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$, if $|x| < 1$

f. $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

g. $(1 + x)^n$, for $|x| < 1$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, |x| < 1$$

OBJECTIVE QUESTIONS

1. Find $\lim_{p \to \infty} \frac{p^5 \cdot p!}{5.6 \ldots (5+p)}$

a. 4 ! b. 5 ! c. 0 d. ∞

2. Find $\lim_{x \to 0} \frac{\sin(x^2)}{x}$

a. ∞ b. -1 c. 0 d. 2^2

3. The value of $\lim_{x \to 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x}$

a. 1 b. -1 c. 0 d. does not exist

4. To verify Rolle's theorem which one is essential?

a. continuous and differentiable in open interval

b. continuous in open interval and differentiable in closed interval

c. continuous in closed interval and differentiable in open interval

d. continuous and differentiable in closed interval

5. When Rolle's theorem is verified for $f(x)$ on $[a, b]$, then there exist c such that

a. $c \in [a, b]$ such that $f'(c) = 0$

b. $c \in (a, b)$ such that $f'(c) = 0$

c. $c \in [a, b)$ such that $f'(c) = 0$

d. $c \in (a, b]$ such that $f'(c) = 0$

6. If the function $f(x) = x^2 - 8x + 12$ satisfies the condition of Rolle's theorem on (2, 6), find the value of c such that $f'(c) = 0$

a. 6 b. 4 c. 8 d. 2

7. If $f(x) = \dfrac{\sin x}{x}$, how many points exists such that $f'(c) = 0$ in the interval $[0, 18\pi]$

a. 8 b. 9 c. 17 d. 18

8. If $f(x) = \ln(10 - x^2)$, $x \in [-3, 3]$, find the point in the interval $[-3, 3]$ where slope of tangent is zero

a. 0

b. 2

c. Rolle's theorem is not applied, because function is not continuous in $[-3, 3]$

d. Rolle's theorem is not applied, because function is not differentiable in $[-3, 3]$

9. Find the point 'c' in the curve $f(x) = x^3 + x^2 + x + 1$ in the interval $[0, 1]$, where slope of a tangent to a curve is equal to slope of line joining (0, 1) and (1, 4)

a. 0.64 b. 0.44 c. 0.54 d. 0.34

10. For the function $f(x) = x^3 + x + 1$, we do not have any Rolle's point. The coordinate axes are transformed by rotating them by 60 degree in anti-clockwise sense. The new Rolle's point is

a. $\sqrt{3}/2$

b. The function can never have a Rolle point

c. $\sqrt{3}$

d. $\sqrt{\dfrac{\sqrt{3}-1}{3}}$

11. Find the point on the curve $y = \log x$, tangent at which is parallel to the chord joining points (1, 0) and $(e, 1)$

a. e b. $e - 1$ c. $1 - e$ d. $-e$

12. If $f(a) = f(b)$ in mean value theorem, then it becomes

a. Leibnitz's theorem

b. Rolle's theorem

c. Taylor's series of a function

d. Cauchy's theorem

13. If $f(x) = \sin x \cos x$ is continuous and differentiable in the interval $(0, x)$, then

a. $1 < \dfrac{\cos x \,.\, \sin x}{x} < \sin 2x$

b. $1 < \dfrac{\cos x \,.\, \sin x}{x} < \cos 2x$

c. $1 < \dfrac{\cos x \,.\, \sin x}{x} < x \cos 2x$

d. $1 < \dfrac{\cos x \,.\, \sin x}{x} < 1 + \cos 2x$

14. On the interval [0, 1], the function $x^{25}(1 - x)^{75}$ takes its maximum value at

a. 0

b. 1/2

c. 1

d. 1/4

15. Find the co-efficient of x^2 in the Taylor's series about $x = 0$ for $f(x) = e^{-x^2}$

a. 1/4

b. – 1

c. 1/2

d. –2

16. $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + \ldots + 100x^{100}$ is polynomial in a real variable x, then $f(x)$ has

a. neither maximum nor minimum

b. only one maximum

c. only one minimum

d. one maximum and one minimum

17. Expansion of function $f(x)$ using Maclaurin's series is

a. $f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$
b. $1+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$

c. $f(0)-\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+...$
d. $f(1)+\frac{x}{1!}f'(1)+\frac{x^2}{2!}f''(1)+\frac{x^3}{3!}f'''(1)+...$

18. The real number x when added to its inverse gives the minimum value of the sum at x equal

a. –2 b. 2 c. 1 d. –1

19. If the function $f(x) = 2x^2 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then 'a' is equal to

a. 1/2 b. 3 c. 1 d. 2

20. The necessary condition for the Maclaurin's expansion to be true for the function $f(x)$ is

a. it should be continuous
b. it should be differentiable
c. it should exists at every point
d. it should be continuous and differentiable both

Answers

1. a	**2.** c	**3.** d	**4.** c
5. b	**6.** b	**7.** d	**8.** a
9. c	**10.** d	**11.** b	**12.** b
13. b	**14.** d	**15.** b	**16.** c
17. a	**18.** c	**19.** c	**20.** d

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. Let $f(x)$ be a continuous and differentiable function on R, then show that between any two real root of $f(x)$, there exists atleast one root of $f'(x)$.

2. Let $f: [a, b] \to R$ be thrice differentiable function such that $f(a) = f(b) = f'(a) = f''(a) = 0$, then prove that there exists $c \in (a, b)$ such that $f'''(c) = 0$.

3. Using mean value theorem, prove that $|\sin x - \sin y| \le |x - y| \; \forall \; x, y \in R$.

4. Let $f: (0, \infty) \to R$ be a differentiable function then prove that, for any $a > 0$, if

$$\lim_{x\to\infty} (af(x) + f'(x)) = L, \text{ then } \lim_{x\to\infty} f(x) = \frac{L}{a}.$$

5. If $a_0 + a_1 + ... + a_n = 0$ where $a_i \in R$, $1 \le i \le n$ then show that $a_0 + 2a_1x + ... + (n + 1)a_n x^n = 0$ has atleast one real root in $(0, 1)$.

6. Using mean value theorem, prove the inequality $\frac{y-x}{y} < \log\left(\frac{y}{x}\right) < \frac{y-x}{x}$ for all $0 < x < y$.

7. Let $f(x) \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ and let $g(x) = \sin x \; \forall \; x \in R$. Show that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$ but $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

8. Let $f : R \to R$ be a differentiable function such that $\left|\dfrac{df(x)}{dx}\right| \le 4$ and $f(0) = 0$ then prove that $f(1) \in [-4, 4]$ and $f(2) \in [-8, 8]$.

9. Let I be an interval and $f : I \to R$ be differentiable on I. Show that if the derivative f' is never 0 on I, then either $f'(x) > 0\ \forall\ x \in I$ or $f'(x) < 0\ \forall\ x \in I$.

10. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle 30° is $\dfrac{4}{81}\,\pi h^3$.

11. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

PRACTICAL

1. Sketch the graph of sine and cosine functions in $[0, 2\pi]$ using MATLAB tool.
2. Plot a graph for e^{3x} on R using MATLAB tool.
3. In M.S. Excel, draw $[x]$, greatest integer function in the interval $[0, 5]$.

ACTIVITY

1. Write a MATLAB code to verify Rolle's theorem for the function
   i. x^2 in the interval $[-3, 3]$
   ii. $(x + 2)^3 * (x - 3)^4$ in the interval $[-2, 3]$.
   Also Plot the curve for the same.

KNOW MORE

1. The value of $\lim\limits_{x \to 0} \left\{\dfrac{a^x + b^x + c^x}{3}\right\}^{1/x}$ is

a. abc b. $(abc)^{1/3}$ c. $(abc)^{1/8}$ d. $\dfrac{1}{abc}$

2. The product of minimum value of x^x and maximum value of $\left(\dfrac{1}{x}\right)^x$ is

a. e b. $\dfrac{1}{e}$ c. 1 d. e^2

3. The expansion of $e^{\sin x}$ is ......

a. $1 + x + \dfrac{x^2}{2} + \dfrac{x^4}{8} + \ldots$ b. $1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8} + \ldots$

c. $1 + x - \dfrac{x^2}{2} + \dfrac{x^4}{8} + \ldots$ d. $1 + x + \dfrac{x^2}{2} - \dfrac{x^5}{10} + \ldots$

4. Find relation between 'a' and 'b' such that the following limit obtain after a single application of L'Hospital Rule on $\underset{x \to 0}{\text{lt.}} \dfrac{ae^x + be^{2x}}{be^x + ae^{2x}}$.

a. $b/a = 2$ b. $a/b = 2$ c. $a = b$ d. $a = -b$

5. Find how many rounds of differentiation are required to have finite limit for

$\underset{x \to 0}{\text{lt.}} \dfrac{\cos(ax) + \cos(bx) - 2\cos(cx)}{\cos(ax) + 2\cos(bx) - 3\cos(cx)}$, given that $a \neq b \neq c$

a. 3 b. 0 c. 2 d. 4

Answers

1. b **2.** c **3.** b **4.** d
5. c

REFERENCES/SUGGESTED READINGS

1. Bary, N.K. (1964). A Treatise on Trigonometric Series, Pergamon Press.
2. Courant R. (1988). Differential and Integral Calculus, Wiley, NewYork.
3. Dass, H.K. Advanced Engineering Mathematics, S. Chand Publications.
4. Fleming, W.H. (1965). Functions of Several Variables, Addison Wesley Publishing Company, Reading, MA.
5. Garg, Reena (2019). Engineering Mathematics-I, 2nd Edition, Khanna Book Publishing Co. (P) Ltd.
6. Grewal, B.S. Higher Engineering Mathematics, Khanna Publishers.
7. Jain, R.K.; Iyengar, S.R.K. Advanced Engineering Mathematics, 2nd Edition, Narosa.
8. Knopp, K. (1947). Theory of Functions, Dover, NewYork.
9. Philip, Franklin (1940). A Treatise on Advanced Calculus, Wiley, Inc. NewYork.
10. Piskunov, N. (1969). Differential and Integral Calculus, MIR Publishers, Moscow.
11. Prasad, Chandrika (1967). Mathematics for Engineers, Pothishala Private Ltd.
12. Ram, Babu. Engineering Mathematics, Pearson.
13. Thomas, G.B. and Finney, R.L.(1992). Calculus and Analytic Geometry, Addision-Wesley, Reading, MA.
14. Tichmarsh, E.C. (1939). Theory of Functions, Oxford University Press, London.

3 Sequences and Series

UNIT SPECIFICS

In this unit we have given a detailed self-explanatory theory of the topics - convergence of sequence and series, tests for convergence, power series, Taylor's series, series for exponential, trigonometric and logarithm functions, Fourier series: half range sine and cosine series, Parseval's theorem, we have given many exercises in the form of objective questions and subjective questions. Bloom's Taxonomy has been adhered to.

RATIONALE

Concepts of sequences and series are very important in mathematics, with their wide range of applications in the area of finance, physics, statistics and engineering. They play a significant role in predicting, evaluating and monitoring the outcome of a situation or an event, which helps a lot in taking decisions for our daily lives. Fourier series is broadly used in telecommunication system, for modulation and demodulation of voice signals; also have a wide significance in the input, output and calculation of pulses and their sine or cosine graph which is very useful to solve various complex problems in different fields of Engineering.

PRE-REQUISITES

1. Knowledge of integration and differentiation.
2. Clear concept about Trigonometric ratios and how to find value of an angle in any quadrant.
3. Knowledge of logarithmic and exponential functions and their graph.
4. Evaluation of limit of a function.

UNIT OUTCOMES

After completion of this unit, students will be able to-

U3-01: Recognize bounded, convergent, divergent, and monotonic sequences; calculate g.l.b., l.u.b. and the limit of a sequence; apply Ratio, Root, Raabe's, Logarithm, Gauss tests on positive term series and Leibnitz's test on alternating series; utilize comparison tests for convergence and absolute convergence of an infinite series.

U3-02: Represent functions as power series; calculate radius of convergence; evaluate the region in which power series converges.

U3-03: Approximate any function whether it is algebraic or transcendental, as a polynomial by using Taylor's and Maclaurin's series.

U3-04: Apply the concept of Fourier series; learn the expansion and representation of functions of one variable by differentiation and integration of Fourier series.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 3 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1-Weak Correlation; 2-Medium Correlation; 3-Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U3-01	2	–	3	–	1
U3-02	–	1	3	1	–
U3-03	–	3	–	–	1
U3-04	1	–	2	1	–

HISTORY

The first known concept of calculating the area under the arc of a parabola with the summation of an infinite series was introduced by the Greek mathematician Archimedes which is used in the area of calculus today. James Gregory (17th century) worked in the new decimal system on infinite series and published several results on Maclaurin series. In 1715, a general method for constructing the Taylor series for all functions for which they exist was provided by Brook Taylor. Leonhard Euler in the 18th century, developed the theory of hyper geometric series and q-series. In 1807 Fourier started to work on the expansion of a given function of x in terms of the sines or cosines of multiples of x. Fourier was the first to assert and attempt to prove the general theorem.

Leohard Euler

INTRODUCTION

If we consider a simple pendulum, in order to count the oscillations, when it moves to and from, sequences are used.

Let us consider a cinema theatre having 30 seats on the first row, 32 seats on the second row, 34 seats on the third row, and so on and has total 40 rows of seats. How many seats are in the theatre?

To solve such type of problems we need to learn sequences and series.

Here, we need to know how many seats are there in the cinema theatre, which means we are counting things and finding a total. In other words, we need to add up all the seats on each row. Since we are adding things up, this can be looked as a series.

3.1 SEQUENCE

A sequence is defined as a function $a : N \to S$, where domain is the set of natural number and range be any non-empty set.

> **Remark:** If range of sequence is any subset of real numbers then that sequence is known as real sequence.

If $a : N \to R$ be a sequence, then image of each $n \in N$, is denoted by a_n. Thus a_1, a_2, a_3 are real numbers associated with 1, 2, 3 by the function 'a' and are called first, second, third terms of the sequence respectively.

The sequence $a : N \to R$ is denoted by $<a_n>$ which when represented in expanded form is written as $<a_1, a_2, a_3, ..., a_n, ...>$.

3.1.1 Methods to Describe a Sequence

A sequence can be described in following ways:

i. By writing first few terms of the sequence, till the pattern of sequence becomes clear.
   For example: $<1, 8, 64, 125, ...>$ is a sequence whose n^{th} term is n^3.

ii. By giving a formula for its n^{th} term.
   For example: $<a_n> = \left\langle \frac{1}{n} \right\rangle$ whose expanded form is $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} ...... \right\rangle$

iii. By giving first few terms of sequence and a relation to determine the other terms of the sequence.
   For example: $a_1 = 3$ and $a_{n+1} = 8 + a_n \ \forall \ n \in N$

$$a_1 = 3, a_2 = 8 + a_1 = 8 + 3 = 11$$
$$a_3 = 8 + a_2 = 8 + 11 = 19,$$
$$a_4 = 8 + a_3 = 8 + 19 = 27$$

   Thus, the sequence is $<3, 11, 19, 27, ...>$.

3.1.2 Range Set of Sequence

The set of all distinct terms of a sequence is called range.

The range of sequence $<a_n>$ is denoted by $\{a_n\}$.

For example: If $a_n = <(-1)^n>$, then

$$<a_n> = <-1, 1, -1, 1, ..., >$$

Range set of sequence is $\{-1, 1\}$.

> **Remarks:**
> i. The terms of a sequence occurring at different positions are treated as distinct terms even if they have the same value.
> ii. The number of terms of sequence is always infinite but range set of sequence may be finite since it contains only distinct terms of the sequence.

3.1.3 Constant Sequence

A sequence $<a_n>$ defined by $a_n = c \ \forall \ n \in N$ is called a constant sequence.

Thus, $<a_n> = <c, c, c, ...>$ is a constant sequence with range $= \{c\}$, a singleton set, a finite set too.

3.1.4 Bounded Sequence

a. **Bounded above sequence:** A sequence $<a_n>$ is said to be bounded above if there exists a real number K such that $a_n \leq K\ \forall\ n \in N$, K is called an upper bound of sequence $<a_n>$.

For example: Sequence $<-n> = <-1, -2, -3, ...>$ is bounded above. -1 is the upper bound of this sequence as $a_n \leq -1\ \forall\ n \in N$.

b. **Bounded below sequence:** A sequence $<a_n>$ is said to be bounded below if there exists a real number k such that $k \leq a_n\ \forall\ n \in N$, k is called the lower bound of the sequence $<a_n>$.

For example: Sequence $<n> = <1, 2, 3, ...>$ is bounded below. 1 is the lower bound of this sequence as $1 \leq a_n\ \forall\ n \in N$.

c. **Bounded sequence:** A sequence $<a_n>$ is said to be bounded if there exists real number k and K such that $k \leq a_n \leq K\ \forall\ n \in N$.

For example: sequence $\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, ... \right\rangle$ is bounded above as well as below.

If we have $0 \leq a_n \leq 1\ \forall\ n \in N$, then 0 is the lower bound and 1 is the upper bound of this sequence.

3.1.5 Unbounded Sequence

A sequence $<a_n>$ is said to be unbounded if it is not bounded *i.e.,* either it is unbounded above or unbounded below or both.

i. Sequence $<(-1)^n . n> = <-1, 2, -3, 4, ...>$ is unbounded having no lower and upper bound.

ii. Sequence $<n> = <1, 2, 3, ...>$ having lower bound but unbounded above

$\therefore$ $<n>$ is unbounded sequence.

3.1.6 Least Upper Bound (Supremum) of Sequence

A real number u is called least upper bound (l.u.b.) or (supremum) of the sequence $<a_n>$ if

i. $a_n \leq u\ \forall\ n \in N$ *i.e.,* u is upper bound of sequence $<a_n>$

ii. If u' is any other real number such that $a_n \leq u'\ \forall\ n \in N$, then $u \leq u'$ *i.e.,* no other upper bound is less than u.

For example:

i. $<a_n> = \left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\rangle$ is bounded above and 1 is upper bound of $<a_n>$ as $a_n \leq 1\ \forall\ n \in N$.

Also, 2, 3, 4, ... are upper bound of $<a_n>$ but 1 is the least upper bound among them.

$\therefore$ 1 is l.u.b./supremum of $<a_n>$

ii. $<a_n> = <n> = <1, 2, 3, ...>$ is unbounded above and hence no l.u.b.

iii. $<a_n> = \left\langle -\frac{1}{n} \right\rangle = \left\langle -1, -\frac{1}{2}, -\frac{1}{3}, ... \right\rangle$ is bounded above and 0 is the upper bound of $<a_n>$ as $a_n \leq 0\ \forall\ n \in N$. Also 1, 2, 3, ... so on, are upper bound of $<a_n>$ but 0 is l.u.b. among them.

$\therefore$ 0 is l.u.b. of $<a_n>$.

> **Note:**
> i. l.u.b. of a sequence may or may not exist.
> ii. l.u.b. of a sequence, if exists, is unique.
> iii. l.u.b. of a sequence may or may not belongs to the sequence.

3.1.7 Greatest Lower Bound (Infimum) of Sequence

A real number l is called the greatest lower bound (g.l.b.) (or infimum) of a sequence $<a_n>$ if

i. $a_n \geq l \ \forall \ n \in N$ *i.e.,* l is the lower bound of the $<a_n>$.

ii. If l' is any other real number such that $l' \leq a_n \ \forall \ n \in N$, then $l \geq l'$ *i.e.,* no other lower bound is greater than l.

For example:

i. $<a_n> = \left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$ is bounded below and 0 is the lower bound of $<a_n>$ as $a_n \geq 0 \ \forall \ n \in N$

Also $-1, -2, -3, \ldots$ so on are lower bounds of $<a_n>$ but 0 is the greatest among them.

$\therefore$ 0 is g.l.b/infimum of $<a_n>$

ii. $<a_n> = <-n> = <-1, -2, -3, \ldots>$ is unbounded below and hence has no g.l.b.

iii. $<a_n> = \left\langle \frac{(-1)^n}{n} \right\rangle = \left\langle -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$ is bounded below and -1 is the lower bound of $<a_n>$ as $a_n \geq -1 \ \forall \ n \in N$.

Also, $-2, -3, -4, \ldots$ so on, are lower bounds of $<a_n>$ but -1 is greatest among them.

$\therefore$ -1 is g.l.b/infimum of $<a_n>$.

> **Note:**
> i. g.l.b of a sequence may or may not exist.
> ii. g.l.b. of a sequence, if exists, is unique
> iii. g.l.b. of a sequence may or may not belongs to the sequence.

3.2 CONVERGENT, DIVERGENT AND OSCILLATORY SEQUENCE

a. **Convergent Sequence:** A sequence $<a_n>$ is said to be convergent to a real number l if for any given $\varepsilon > 0$, however small it may be, there exists a positive integer m (depending upon ε) such that $| a_n - l | < \in \ \forall \ n \geq m$.

The real number l is called limit of the sequence $<a_n>$.

OR

A sequence $<a_n>$ is said to be convergent if $\lim_{n \to \infty} a_n$ is finite say 'l'.

The real number l is called the limit of sequence $<a_n>$ and is written as $\lim_{n \to \infty} a_n = l$ or $a_n \to l$ as $n \to \infty$.

b. **Divergent Sequence:** Any sequence is said to be divergent if it is not convergent.

i.e., $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$

c. **Oscillatory Sequence:** If a sequence $\langle a_n \rangle$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$ then it is called oscillating sequence.

Oscillatory sequence are of two types:

a. **Oscillate Finitely:** A sequence is said to oscillate finitely if
   i. it is bounded
   ii. it neither converges nor diverges.

b. **Oscillate Infinitely:**
   i. it is unbounded
   ii. it neither converges nor diverges

For example:

1. $\langle(-1)^n\rangle = \langle -1, 1, -1, 1, \ldots\rangle$ oscillates finitely.
2. $\langle 1 + (-1)^n\rangle = \langle 0, 2, 0, 2 \ldots\rangle$ oscillates finitely.
3. $\langle(-1)^n . n\rangle = \langle -1, 2, -3, 4, -5, 6 \ldots\rangle$ oscillates infinitely.
4. $\left\langle 1, 2, \frac{1}{2}, 3, \frac{1}{3} \ldots \right\rangle$ oscillates infinitely.

3.2.1 Limit of a Sequence

A sequence $\langle a_n \rangle$ is said to approach the limit l when $n \to \infty$, if for each $\varepsilon > 0$ there exist a positive integer m (depending upon ε) such that $|a_n - l| < \varepsilon \ \forall\ n \geq m$. *i.e.* $a_n \in (l - \varepsilon, l + \varepsilon)$.

Symbolically we can write $\lim_{n\to\infty} a_n = l$.

For example:

1. $\left\langle \frac{1}{n} \right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3} \ldots \right\rangle$ has only 0 as its limit.
2. $\langle n^2 \rangle = \langle 1, 4, 9, \ldots\rangle$ has no limit.

3.2.2 Limit Point of a Sequence

A real number l is said to be limit point of a sequence $\langle a_n \rangle$, if for a given $\varepsilon > 0$ and for a given positive integer m, there exists a positive integer $K > m$ such that $|a_K - l| < \in$ *i.e.*, $a_K \in (l - \varepsilon, l + \varepsilon)$.

Difference between Limit and Limit Point

The main difference between limit and limit point is that, in case of limit, the number of terms that lie inside the interval $(l - \varepsilon, l + \varepsilon)$ must be infinite but the number of terms that lie outside this interval must be finite; whereas in case of limit point, the number of terms that lie inside the interval $(l - \varepsilon, l + \varepsilon)$ must be infinite but number of terms that lie outside this interval may be finite or infinite.

Important Observations

1. Limit of a sequence is also the limit point of the sequence but converse is not true.

   For example: Consider the sequence $\langle a_n \rangle$ where $a_n = (-1)^n$

   The sequence is $\langle -1, 1, -1, 1 \ldots\rangle$ and $-1 \leq a_n \leq 1$ for all n.

Limit point of sequence is $\{-1, 1\}$

But this sequence has no limit.

2. Every convergent sequence has a unique limit therefore unique limit point, but if a sequence has a unique limit point, it may not be convergent.

   For example: Consider the sequence $<a_n>$ where $a_n = <1, 2, 1, 3, 1, 4, ...>$ limit point of sequence is 1 but this sequence is not convergent.

3. Every convergent sequence is bounded but converse is not true.

   For example: Consider the sequence $<a_n>$ where $a_n = (-1)^n$

   The sequence is $<-1, 1, -1, 1, ...>$ and $-1 \le a_n \le 1 \ \forall \ n \in N$.

   Clearly, the sequence $<a_n> = <(-1)^n>$ is bounded but not converges to a limit.

4. A bounded sequence either converges or oscillates finitely.
5. An unbounded sequence either diverges to $+\infty$ or $-\infty$ or oscillates infinitely.

3.2.3 Null Sequence

A sequence $<a_n>$ is said to be null sequence if it converges to 0. *i.e.*,

$$\lim_{n \to \infty} a_n = 0$$

For example: The sequence $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{2n} \right\rangle, \left\langle \frac{(-1)^n}{n} \right\rangle$ are null sequence.

Results of Null Sequences

1. A sequence $<a_n>$ converges to l iff the sequence $<a_n - l>$ is a null sequence.
2. If $<a_n>$ is a sequence, then $a_n \to 0$ iff $|a_n| \to 0$
3. Sum of null sequences is also a null sequence.

$$\lim_{n \to \infty} a_n = 0, \ \lim_{n \to \infty} b_n = 0 \Rightarrow \lim_{n \to \infty} (a_n + b_n) = 0$$

Results on Inequalities and Comparisons

1. If a sequence $<a_n>$ converges to a and $a_n \ge 0 \ \forall \ n$, then $a \ge 0$.
2. If $\lim\limits_{n \to \infty} a_n = a$, $\lim\limits_{n \to \infty} b_n = b$ and $a_n \le b_n \ \forall \ n$, then $a \le b$.
3. If $\lim\limits_{n \to \infty} a_n = a$ and $a_n \ge k \ \forall \ n$, then $a \ge k$.

3.2.4 Algebra of Limits

1. If $\lim\limits_{n \to \infty} a_n = a$ then $\lim\limits_{n \to \infty} K a_n = Ka$, where $K \in \mathbb{R}$
2. If $\lim\limits_{n \to \infty} a_n = a$ then $\lim\limits_{n \to \infty} |a_n| = |a|$
3. If $\lim\limits_{n \to \infty} a_n = a$ and $\lim\limits_{n \to \infty} b_n = b$, then

a. $\lim_{n\to\infty} (a_n + b_n) = a + b$

b. $\lim_{n\to\infty} (a_n . b_n) = ab$

c. $\lim_{n\to\infty} \left(\frac{1}{a_n}\right) = \frac{1}{a}$ provided $a_n \neq 0 \ \forall \ n, a \neq 0$

d. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ provided $b_n \neq 0 \ \forall \ n, b \neq 0$

Converse of the above result, however, not true.

S. No.	$<a_n>$	$<b_n>$	$<a_n + b_n>$
1.	Converges	Diverges to ∞	Diverges to ∞
2.	Converges	Diverges to $-\infty$	Diverges to $-\infty$
3.	Diverges to ∞	Diverges to ∞	Diverges to ∞
4.	Diverges to $-\infty$	Diverges to $-\infty$	Diverges to $-\infty$

S. No.	$<a_n>$	$<b_n>$	$<a_n . b_n>$
1.	Diverges to $+\infty$	Diverges to ∞	Diverges to ∞
2.	Diverges to ∞	Diverges to $-\infty$	Diverges to $-\infty$

SOME SOLVED EXAMPLES

Example 3.1. *Discuss boundedness of the following sequence $<a_n>$ where a_n is given by*

i. $a_n = 5$

ii. $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2}$

Solution. i. Here $a_n = 5$ is a constant sequence *i.e.*, $<a_n> = <5, 5, 5, \ldots>$

Clearly, $5 \leq a_n \leq 5 \ \forall \ n \in N$

Thus, sequence is bounded.

ii.
$$a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2}$$
$$< \frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2} \qquad \left[\because \ n^2 < (n+1)^2 \text{ so, } \frac{1}{n^2} > \frac{1}{(n+1)^2}\right]$$
$$= \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \leq 1 + 1 = 2 \ \forall \ n$$

Also, $a_n > 0 \ \forall \ n$

Thus, $0 < a_n < 2 \ \forall \ n \in N$

Hence, the sequence $<a_n>$ is bounded.

Example 3.2. *For the given sequences, discuss the following properties: boundedness, l.u.b., g.l.b., limit points, convergence, divergence, oscillating finitely or infinitely.*

i. $a_n = 1 + (-1)^n$ ii. $a_n =$ *least prime divisor of* n^2, $n \geq 2$

Solution. i. $$a_n = 1 + (-1)^n = \begin{cases} 2, n \text{ is even} \\ 0, n \text{ is odd} \end{cases}$$

$\Rightarrow$ $<a_n> = <0, 2, 0, 2, 0, ...>$

Clearly $0 \leq a_n \leq 2 \; \forall \; n \in N$

So, limit point of sequence = {0, 2}

Thus sequence $<a_n>$ is bounded and l.u.b = 2 and g.l.b = 0

Given sequence not converges to a point rather this sequence is oscillating finitely.

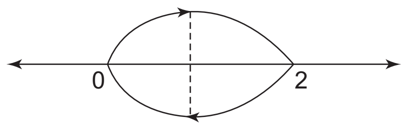


Fig. 3.1

sequence is oscillating between 0 and 2.

ii. $a_n =$ least prime divisor of n^2, $n \geq 2$, so, the sequence

$<a_n> = <2, 3, 2, 5, 2, ..>$

Clearly $2 \leq a_n \; \forall \; n \in N$

But there does not exist any K such that $a_n \leq K \; \forall \; n \in N$

Thus, $<a_n>$ is bounded below and unbounded above with l.u.b. = does not exist

g.l.b. = 2

Limit point of sequence $<a_n>$ = {2, 3, 5, ..} *i.e.*, {p; p is prime}

This sequence does not converges to a limit and also not diverges to $\pm \infty$ although it oscillates infinitely.

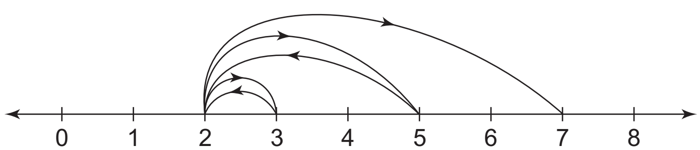


Fig. 3.2

Example 3.3. *By definition, show that*

i. $\lim\limits_{n \to \infty} \dfrac{3n+7}{4n+8} = \dfrac{3}{4}$ ii. *<n> diverges to* ∞

Solution. i. Consider $a_n = \dfrac{3n+7}{4n+8}$

Let $\varepsilon > 0$ be given real number

Now, $$\left| a_n - \frac{3}{4} \right| = \left| \frac{3n+7}{4n+8} - \frac{3}{4} \right|$$

$$= \left| \frac{12n + 28 - 12n - 24}{4(4n+8)} \right|$$

$$\left| a_n - \frac{3}{4} \right| = \frac{1}{4n+8} < \frac{1}{4n}$$

$$\therefore \quad \left| a_n - \frac{3}{4} \right| < \varepsilon \text{ if } \frac{1}{4n} < \varepsilon$$

$$\Rightarrow \quad \left| a_n - \frac{3}{4} \right| < \varepsilon \text{ if } n > \frac{1}{4\varepsilon}$$

$$\Rightarrow \quad \left| a_n - \frac{3}{4} \right| < \varepsilon \text{ if } n > m$$

where m is a positive integer $> \frac{1}{4\varepsilon}$

$$\Rightarrow \quad \left| a_n - \frac{3}{4} \right| < \varepsilon \ \forall \ n \geq m$$

$$\therefore \quad \lim_{n \to \infty} a_n = \frac{3}{4}$$

Thus, the sequence $<a_n> = \left\langle \frac{3n+7}{4n+8} \right\rangle$ converges to $\frac{3}{4}$.

Alter Method:

Given, $$a_n = \frac{3n+7}{4n+8}$$

Dividing numerator and denominator by n.

$$a_n = \frac{3 + \frac{7}{n}}{4 + \frac{8}{n}}$$

Taking $\lim_{n \to \infty}$ on both sides, we get

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3 + \frac{7}{n}}{4 + \frac{8}{n}}$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{3n+7}{4n+8} = \frac{3}{4}$$

Thus, the sequence $\left\langle \frac{3n+7}{4n+8} \right\rangle$ converges to $\frac{3}{4}$.

ii. Let $a_n = n$ and $k > 0$ be any given number.

Now $$a_n > k \text{ if } n > k$$

$$a_n > k \text{ if } n \geq m, \text{ where } m \text{ is least positive integer} > k$$

$\therefore \quad a_n > k \ \forall \ n \geq m$

$\Rightarrow \quad \lim_{n \to \infty} a_n = \infty$

$\therefore \quad <a_n> = <n>$ diverges to ∞

Alternate Method:

$$<a_n> = <n> = <1, 2, 3, 4, ...>$$

Clearly, this sequence is bounded below and unbounded above.

There does not exist any $k \in R$ such that $a_n \leq k \ \forall \ n \in N$

$\therefore \quad \lim_{n \to \infty} a_n = \infty$

$\therefore \quad <a_n> = <n>$ diverges to ∞.

Example 3.4. *Give examples of sequences $<a_n>$ and $<b_n>$ such that*

i. *$<a_n> \to \infty$, $<b_n> \to -\infty$ but $<a_n + b_n>$ converges.*

ii. *$<a_n> \to \infty$, $<b_n> \to -\infty$ but $<a_n + b_n>$ oscillates finitely.*

iii. *$<a_n>$ converges and $<b_n> \to \infty$ but $<a_n . b_n>$ diverges to ∞.*

iv. *$<a_n>$ converges and $<b_n> \to \infty$ but $<a_n . b_n>$ oscillates finitely.*

v. *$<a_n - b_n>$ convergent but $<a_n>$ and $<b_n>$ both are not convergent.*

Solution. i. Let $a_n = n^2$ and $b_n = -n^2 \ \forall \ n$

$\therefore \quad <a_n> \to \infty$ and $<b_n> \to -\infty$

Now, $\quad a_n + b_n = n^2 - n^2 = 0 \ \forall \ n$

$\Rightarrow \quad <a_n + b_n> \to 0$

Thus sequence $<a_n + b_n>$ converges to 0.

ii. Let $\quad a_n = n$ and $b_n = \begin{cases} -n, & n \text{ is even} \\ -n+1, & n \text{ is odd} \end{cases}$

$\therefore \quad <a_n> = <1, 2, 3, 4, ...> \to \infty$

and $\quad <b_n> = <0, -2, -2, -4, -4, ...> \to -\infty$

Now, $<a_n + b_n> = <1, 0, 1, 0, 1, ...>$ has two limit points $\{1, 0\}$ and oscillates finitely between 0 and 1.

iii. Let $a_n = -1$ and $b_n = -n \ \forall \ n$

$<a_n>$ is a constant sequence and converges to -1

$$<b_n> = <-n> = <-1, -2, -3, ...> \to -\infty$$

Now $\quad a_n . b_n = (-1)(-n) = n$

$\Rightarrow \quad <a_n . b_n> = <n> \to \infty$

Thus the sequence $<a_n . b_n>$ diverges to ∞.

iv. Let $a_n = \frac{(-1)^n}{n}$, $b_n = n^2 \ \forall \ n$

$\therefore \quad <a_n> = \left\langle \frac{(-1)^n}{n} \right\rangle \to 0$ and $<b_n> = <n^2> \to \infty$

Now, $\quad a_n . b_n = \frac{(-1)^n}{n} . n^2 = (-1)^n . n$

$$\Rightarrow \quad <a_n \,.\, b_n> = <(-1)^n \,.\, n>$$
$$= <-1, 2, -3, 4, -5, ...>$$

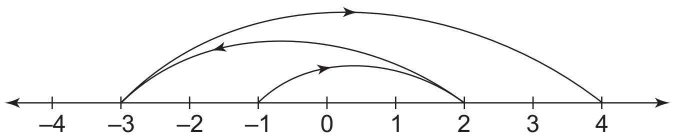


Fig. 3.3

Thus, $<a_n \,.\, b_n>$ is unbounded and oscillates infinitely.

v. Let $\quad a_n = n$ and $b_n = n \; \forall \; n$

$\therefore \quad a_n - b_n = 0 \; \forall \; n$

$\therefore \quad <a_n - b_n>$ is convergent and converges to 0 but neither $<a_n>$ nor $<b_n>$ is convergent.

Example 3.5. *Give an example of a sequence*

i. *which is bounded and oscillates finitely.*

ii. *having no limit point.*

Solution. i. Consider the sequence $<a_n>$ where $a_n = (-1)^n$

$\therefore \quad <a_n> = <-1, 1, -1, 1, ...>$

$\Rightarrow \quad -1 \le a_n \le 1 \; \forall \; n \in N$

$\Rightarrow \quad a_n$ is bounded and oscillates between –1 and 1

ii. Consider the sequence $<a_n>$ where $a_n = n$

$\therefore \quad <n> = <1, 2, 3, 4, ...>$

Sequence $<n>$ is diverges to ∞ and have no limit point.

EXERCISE 3.1

1. Discuss the boundedness of the following sequences $<a_n>$ where

i. $a_n = C$ (any constant)

ii. $a_n = (-1)^n \, 7$

iii. $a_n = n^2$

iv. $a_n = \dfrac{n}{n^2+1}$

v. $a_n = n^{\text{th}}$ prime

2. For the given sequences, discuss the following properties: boundedness, l.u.b., g.l.b., limit points, convergence, divergence, oscillating finitely or infinitely.

i. $a_n = \dfrac{1}{2^n}$

ii. $a_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$

iii. $a_n = -n^2$

iv. $a_n = \dfrac{-1}{2n+1}$

v. $a_n = <1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5 ...>$

vi. $a_n =$ Unit digit of $(214)^n$

vii. $a_n = \begin{cases} 0, & \text{if } n = 1 \text{ or prime} \\ n, & \text{if } n \text{ is composite} \end{cases}$

viii. $a_n = \begin{cases} 3, & \text{if } n \text{ is odd} \\ 1/n, & \text{if } n \text{ is even} \end{cases}$

3. By definition, show that

i. the sequence $\left\langle \frac{1}{3^n} \right\rangle$ converges to 0.

ii. the sequence $\left\langle \frac{n^2+1}{2n^2+5} \right\rangle$ converges to 1/2

iii. the sequence $<-n^2>$ diverges to $-\infty$

iv. the sequence $\left\langle \frac{2n+3}{3n+4} \right\rangle$ converges to 2/3

v. the sequence $<n^2>$ diverges to ∞.

4. By definition, show that

i. $\lim\limits_{n\to\infty} \frac{n^2+1}{3n^2+2} = \frac{1}{3}$

ii. $\lim\limits_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$

iii. $\lim\limits_{n\to\infty} \frac{3n-1}{4n+5} = \frac{3}{4}$

iv. $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$

v. $\lim\limits_{n\to\infty} \frac{n^2+3n+5}{2n^2+5n+7} = \frac{1}{2}$

5. Give examples of sequences $<a_n>$ and $<b_n>$ such that

i. $<a_n> \to \infty$, $<b_n> \to -\infty$ but $<a_n + b_n>$ diverges to ∞

ii. $<a_n> \to \infty$, $<b_n> \to -\infty$ but $<a_n + b_n>$ diverges to $-\infty$

iii. $<a_n> \to \infty$, $<b_n> \to -\infty$ but $<a_n + b_n>$ oscillates infinitely

iv. $<a_n>$ converges and $<b_n> \to \infty$ but $<a_n . b_n>$ converges

v. $<a_n>$ converges and $<b_n> \to \infty$ but $<a_n . b_n>$ diverges to $-\infty$

vi. $<a_n>$ converges and $<b_n> \to \infty$ but $<a_n . b_n>$ oscillates finitely.

vii. $<a_n>$ converges and $<b_n> \to -\infty$ but $<a_n . b_n>$ oscillates infinitely.

viii. $<a_n>$ converges and $<b_n> \to -\infty$ but $<a_n . b_n>$ converges.

ix. $<a_n>$ converges and $<b_n> \to \infty$ but $<a_n . b_n>$ oscillates finitely.

x. $<a_n . b_n>$ convergent but $<a_n>$ and $<b_n>$ not convergent.

xi. $\left\langle \frac{a_n}{b_n} \right\rangle$ convergent but $<a_n>$ and $<b_n>$ both are not convergent.

6. Given an example of the sequence

i. which is unbounded and oscillates infinitely

ii. having two limit points

7. i. Show that $\left\langle \frac{\sin n\frac{\pi}{3}}{\sqrt{n}} \right\rangle$ is a null sequence.

ii. Show that $\left\langle \frac{n!}{n^n} \right\rangle$ is a null sequence.

Hints: Use result of inequalities and comparison,

$$a_n = \frac{n!}{n^n},\ b_n = \frac{1}{n}$$

$$\Rightarrow \qquad |a_n| \le |b_n| \ \forall\ n$$

Answers

1. i. Bounded ii. Bounded iii. Unbounded iv. Bounded
v. Unbounded

2. i. Bounded, l.u.b. = 1/2, g.l.b. = 0, limit point = {0}, converges to 0.
ii. Bounded, l.u.b. = 3/2, g.l.b. = –2, limit point = {–1, 1}, oscillates finitely
iii. Unbounded, l.u.b. = –1, g.l.b. = No, no limit point, diverges to $-\infty$.
iv. Bounded, l.u.b. = 0, g.l.b. = –1/3, limit point = {0}, converges to 0.
v. Unbounded, l.u.b. = does not exist, g.l.b. = 1, limit point = $\{n, n \in N\}$, oscillates infinitely
vi. Bounded, l.u.b. = 6, g.l.b. = 4, limit point = {4, 6}, oscillates finitely.
vii. Unbounded, l.u.b. = does not exist, g.l.b. = 0, limit point = {0}, oscillates infinitely
viii. Bounded, l.u.b. = 3, g.l.b. = 0, limit point = {0, 3}, oscillates infinitely

5. i. $a_n = 2n, b_n = -n$ ii. $a_n = n, b_n = -2n$

iii. $b_n = -n, a_n = \begin{cases} n^2, & n \text{ is odd} \\ n+1, & n \text{ is even} \end{cases}$ iv. $a_n = \dfrac{1}{n^2}, b_n = n$

v. $a_n = -\dfrac{1}{n}, b_n = n^2$ vi. $a_n = \dfrac{(-1)^n}{n^2}, b_n = n$

vii. $a_n = \dfrac{(-1)^n}{n}, b_n = -n^2$ viii. $a_n = \dfrac{1}{n^2}, b_n = -n$

ix. $a_n = \dfrac{(-1)^n}{n^2}, b_n = -n$ x. $a_n = b_n = (-1)^n$

xi. $a_n = b_n = n$

6. i. $<(-1)^n . n>$ ii. $<(-1)^n>$

3.2.5 Monotonic Sequence

A sequence is said to be monotonic if it is either monotonically increasing or decreasing.

a. A sequence $<a_n>$ is said to be monotonically increasing if $a_{n+1} \geq a_n \ \forall \ n \in N$.
i.e., $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$

b. A sequence $<a_n>$ is said to be monotonically decreasing if $a_{n+1} \leq a_n \ \forall \ n \in N$.
i.e., $a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots$

c. A sequence $<a_n>$ is said to be strictly monotonically increasing if $a_{n+1} > a_n \ \forall \ n \in N$.

d. A sequence $<a_n>$ is said to be strictly monotonically decreasing if $a_{n+1} < a_n \ \forall \ n \in N$.

For example:

1. The sequence <1, 1, 2, 2, 3, 3, 4, 4 ...> is a monotonically increasing sequence.
2. The sequence $<n>, <n^2>, <3^n>$ are strictly monotonically increasing.
3. The sequence $\left\langle 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3} \ldots \right\rangle$ is a monotonically decreasing sequence.
4. The sequence $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle$ are strictly monotonically decreasing sequences.

Results of Monotonic Sequences

1. Every monotonically increasing sequence which is bounded above converges to its least upper bound (l.u.b.).
2. Every monotonically decreasing sequence which is bounded below converges to its greatest lower bound (g.l.b.).
3. Every bounded monotonic sequence is convergent.
4. Every monotonically increasing sequence unbounded above diverges to ∞.
5. Every monotonically decreasing sequence unbounded below diverges to –∞.
6. Every monotonically sequence is either convergent or divergent *i.e.,* a monotonic sequence never oscillates.

SOME SOLVED EXAMPLES

Example 3.6. *Prove that the sequence whose nth term is given monotonic. Is the sequence increasing or decreasing?*

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}$$

Solution.
$$a_n = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}$$

$$a_{n+1} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\frac{1}{2^n}+\frac{1}{2^{n+1}}$$

$$a_{n+1}-a_n = \frac{1}{2^{n+1}} > 0 \;\forall\; n$$

$$a_{n+1} > a_n \text{ for all } n$$

Hence, sequence $<a_n>$ is monotonically increasing. **Proved.**

Example 3.7. *Prove that the sequence $<a_n>$ defined by $a_{n+1} = \sqrt{3a_n}$, $a_1 = 1$ converges to 3.*

Solution. The sequence $<a_n>$ in the expanded form is $<1, \sqrt{3}, \sqrt{3\sqrt{3}}, \ldots>$

By using the principle of mathematical induction, we shall prove that the sequence $<a_n>$ is monotonically increasing and bounded above.

Now $\sqrt{3} > 1 \quad\Rightarrow\quad a_2 > a_1$

Assume as our induction hypothesis, that $a_n > a_{n-1}$

$\Rightarrow \quad \sqrt{3a_n} > \sqrt{3a_{n-1}}$

$\Rightarrow \quad a_{n+1} > a_n$

$\Rightarrow$ $<a_n>$ is monotonically increasing by the principle of mathematical induction.

Now $a_1 = 1 < 3$

As our induction hypothesis,

We assume that $a_n < 3$

$\Rightarrow \quad \sqrt{3.a_n} < \sqrt{3.3}$

$\Rightarrow \quad a_{n+1} < 3$

$\Rightarrow$ $<a_n>$ is bounded above by the principle of mathematical induction, so the sequence $<a_n>$ is convergent and let $\lim\limits_{n\to\infty} a_n = l$.

Now, given that $\sqrt{3a_n} = a_{n+1}$

$$\Rightarrow \qquad a_{n+1}^2 = 3a_n$$

$$\Rightarrow \qquad \lim_{n\to\infty} . a_{n+1}^2 = 3 \lim_{n\to\infty} a_n$$

$$\text{or} \qquad l^2 = 3l$$

$$\text{or} \qquad l(l-3) = 0$$

$$l = 0, 3$$

The sequence $<a_n>$ is monotonically increasing and $a_1 = 1$, so

$$a_n > a_1 \ \forall\ n$$

$$\Rightarrow \qquad a_n > 1$$

$$\Rightarrow \qquad \lim_{n\to\infty} a_n \geq 1$$

$$\Rightarrow \qquad l \geq 1$$

$$\therefore \qquad l \neq 0 \text{ and so } l = 3$$

Thus, the sequence $<a_n>$ converges to 3.

EXERCISE 3.2

1. Show that the sequence $<a_n>$ where $a_{n+1} = \dfrac{3+2a_n}{2+a_n}$, $a_1 = 1$ converges. Find its limit.

2. Show that the sequence $<a_n>$ defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$ converges to 2.

3. i. Prove that the sequence $<a_n>$ defined by $a_1 = 1$ and $a_n = \sqrt{2 + a_{n-1}}$ converges to 2.

ii. Prove that the sequence $<a_n>$ defined by $a_{n+1} = \sqrt{7a_n}$, $a_1 = 1$ converges to 7.

4. Discuss the convergence of the sequence $<a_n>$ where

i. $a_n = \dfrac{2n-7}{3n+2}$

ii. $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots \dfrac{1}{3^n}$

Answers

1. $\sqrt{3}$

3.2.6 Squeeze Principle

Statement: If $<a_n>$, $<b_n>$ and $<c_n>$ are three sequences such that

i. $a_n \leq c_n \leq b_n$ for all n

ii. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = l$, then $<c_n>$ also converges to l *i.e.,* $\lim\limits_{n\to\infty} c_n = l$.

3.2.7 Cauchy's First Theorem on Limit

If a sequence $\langle a_n \rangle$ converges to l *i.e.*, $\lim\limits_{n \to \infty} a_n = a$ and $b_n = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$, then the sequence $\langle b_n \rangle$ also converges to 'a'.

> **Remark:** The converse of Cauchy's First theorem on limits is not always true.

For example: Let $a_n = (-1)^n$, then

$$b_n = \frac{a_1 + a_2 + \ldots + a_n}{n} = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{1}{n}, & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow \qquad \lim_{n \to \infty} b_n = 0 \quad \Rightarrow \quad \langle b_n \rangle \text{ converges}$$

but $\langle a_n \rangle$ is not convergent.

Col. If $\lim\limits_{n \to \infty} a_n = a > 0$ where $a_n \geq 0 \ \forall \ n$

i.e., if $\langle a_n \rangle$ is a sequence of positive terms converging to $a > 0$ and $b_n = (a_1 a_2 \ldots a_n)^{1/n}$, then $\langle b_n \rangle$ converges to a.

3.2.8 Cauchy's Second Theorem on Limits

Statement: If $\langle a_n \rangle$ is a sequence of positive terms and $\lim\limits_{n \to \infty} (a_n)^{1/n}$ and $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n}$ both exist finitely or infinitely, then $\lim\limits_{n \to \infty} (a_n)^{1/n} = \lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n}$

> **Some Important Results**
>
> a. Let $\langle a_n \rangle$ be a sequence such that $a_n \neq 0$ for all $n \in N$ and $\dfrac{a_{n+1}}{a_n} \to l$. If $|\,l\,| < 1$, then
> $$\lim_{n \to \infty} a_n = 0$$
>
> b. Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two sequences satisfying the conditions
>
> i. y_n is such that $\langle y_n \rangle$ is monotonically increasing and $y_n \to \infty$ as $n \to \infty$
>
> ii. $\lim\limits_{n \to \infty} \dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} = l$, then
> $$\lim_{n \to \infty} \frac{x_n}{y_n} = l.$$

> **Note:** [(b) known as Stolz theorem.]

SOME SOLVED EXAMPLES

Example 3.8. *Using squeeze principle, show that,*

$$\lim_{n \to \infty} \left[\frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + \ldots + \frac{1}{\sqrt{n^2 + n}} \right] = 1.$$

Solution. Let

$$
\begin{aligned}
a_n &= \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \\
&\geq \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} \\
&= \frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \qquad \dots(1)
\end{aligned}
$$

Again

$$
\begin{aligned}
a_n &= \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \\
&\leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} \\
&= \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \qquad \dots(2)
\end{aligned}
$$

By (1) and (2), we have

$$\frac{1}{\sqrt{1+\frac{1}{n}}} \leq a_n \leq \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Since $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n}}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$, so by squeeze principle,

$$\lim_{n\to\infty} a_n = 1$$

i.e.,
$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right] = 1$$

Example 3.9. *If* $a_n = \dfrac{n!}{n^n}$*, then* $\lim\limits_{n\to\infty} a_n = 0$.

Solution. Here,
$$a_n = \frac{n!}{n^n},\ a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$$

$$
\begin{aligned}
\frac{a_n}{a_{n+1}} &= \frac{n!\,(n+1)^{n+1}}{n^n\,(n+1)!} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n \\
&= \left(1+\frac{1}{n}\right)^n
\end{aligned}
$$

$$\Rightarrow \qquad \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\Rightarrow \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1 \qquad (\because \quad e = 2.718 \text{ approx.})$$

$$\Rightarrow \qquad \lim_{n\to\infty} a_n = 0$$

Example 3.10. *Using Cauchy's first theorem on limits, show that*

$$\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) = 0.$$

Solution. Let $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$

$\therefore$ By Cauchy's first theorem on limits,

$$\lim_{n\to\infty}\left(\frac{a_1+a_2+\ldots+a_n}{n}\right) = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) = 0$$

Example 3.11. *Using Cauchy's second theorem on limits, prove that*

$$\lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n} = e$$

Solution. Let
$$a_n = \left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n$$

then
$$a_{n+1} = \left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\left(\frac{n+2}{n+1}\right)^{n+1}$$

and
$$\frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} = e$$

So, by Cauchy's second theorem on limits,

$$\lim_{n\to\infty} (a_n)^{1/n} = e$$

i.e.,
$$\lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\right] = e$$

Example 3.12. *Show that* $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\right) = 2$

Solution. [**Hints:** Use Result (*b*) Stolz theorem]

Example 3.13. *Evaluate* $\lim\limits_{n \to \infty}\left(1-\dfrac{1}{n^2}\right)^{-n}$

Solution. Let
$$y = \lim_{n \to \infty}\left(1-\frac{1}{n^2}\right)^{-n}$$

Taking log on both sides, we get

$$\log y = \lim_{n \to \infty} \log\left[\left(1-\frac{1}{n^2}\right)^{-n}\right]$$
$$= \lim_{n \to \infty}\left[-n\log\left(1-\frac{1}{n^2}\right)\right]$$
$$= \lim_{n \to \infty}\left[-n\left(-\frac{1}{n^2}-\frac{1}{2n^4}-\frac{1}{3n^6}\ldots\ldots\right)\right]$$

or
$$= \lim_{n \to \infty}\left[\frac{1}{n}+\frac{1}{2n^3}+\frac{1}{3n^5}+\ldots\right] = 0$$

$\Rightarrow$
$$y = 1 \quad \Rightarrow \quad \lim_{n \to \infty}\left(1-\frac{1}{n^2}\right)^{-n} = 1$$

Example 3.14. *Prove that* $\lim\limits_{n \to \infty}\left(\dfrac{n^n}{n!}\right)^{1/n} = e.$

Solution. Let
$$a_n = \left(\frac{n^n}{n!}\right)$$

$\therefore$
$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

Now,
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}$$
$$= \frac{1}{n+1}\cdot\frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)(n+1)^n}{(n+1).n^n}$$
$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$\therefore$
$$\lim_{n \to \infty}\frac{a_{n+1}}{a_n} = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = e$$

Here $<a_n>$ is a sequence of positive terms such that
$$\lim_{n \to \infty}\frac{a_{n+1}}{a_n} = e$$

$\therefore$ By Cauchy's second theorem on limits,

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e$$

$$\therefore \qquad \lim_{n\to\infty} (a_n)^{1/n} = e$$

i.e.,
$$\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n} = e$$

EXERCISE 3.3

1. Using squeeze principle, show that

i. $\lim_{n\to\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2}\right] = 0$

ii. $\lim_{n\to\infty} \left[\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \dots + \frac{1}{\sqrt{2n}}\right] = \infty$

iii. $\lim_{n\to\infty} \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+n)^2}\right] = 0$

iv. $\lim_{n\to\infty} \left[\frac{1}{n^2+1} + \frac{2}{n^2+2} + \dots + \frac{n}{n^2+n}\right] = \frac{1}{2}$

2. Using Cauchy's first theorem on limits, show that

i. $\lim_{n\to\infty} \dfrac{1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}}{n} = 0$

ii. $\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}\right) = 0$

iii. $\lim_{n\to\infty} \frac{1}{n}\left(\frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots + \frac{n+1}{n}\right) = 1$

3. Using Cauchy's second theorem on limits, prove that

i. $\lim_{n\to\infty} (n)^{1/n} = 1$

ii. $\lim_{n\to\infty} (n!)^{1/n} = \infty$

iii. $\lim_{n\to\infty} \left[\frac{(3n)!}{(n!)^3}\right]^{1/n} = 27$

iv. $\lim_{n\to\infty} \left[\frac{(n+1)(n+2)\dots(n+n)}{n^n}\right]^{1/n} = \frac{4}{e}$

4. Show that $\lim_{n\to\infty} \frac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \dots + n^{1/n}) = 1$

5. Evaluate the following limits:

i. $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{3n}$

ii. $\lim_{n\to\infty} \left(1 + \frac{1}{n+4}\right)^{n+5}$

iii. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{n}$

iv. $\lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^{-n}$

Answers

5. i. $e^{3/2}$ ii. e iii. e^{-1} iv. 1

Introduction

The concept of infinite series with a lay-man point of view who knows the addition of number is as follows:

We consider the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\ldots\ldots \tag{1}$$

If a lay-man is asked to find the sum of the above series using calculator or any other device and express the sum in decimal representation, then he will start adding the term with an expectation of finding the exact sum. In this process he will obtain:

Sum of two terms	= 1 + 0.5	= 1.5
Sum of three terms	= 1.5 + 0.25	= 1.75
Sum of four terms	= 1.75 + 0.125	= 1.875
Sum of five terms	= 1.875 + 0.0625	= 1.9375
Sum of six terms	= 1.9375 + 0.03125	= 1.96875
Sum of seven terms	= 1.96875 + 0.015625	= 1.984375

After adding few terms of this series, he will soon come to the conclusion that sum of this series can not be found because the process of adding the terms will never have an end as there are infinite many terms.

Thus we have concluded that it is impossible to find the exact sum of infinitely many terms. Yes, we are right! But keeping this conclusion on one side for a moment, let us restart the process of adding the terms of the series (1). After adding ten terms of the series, we shall obtain the sum 1.998046875. Looking at this number, it seems that after adding one or two of few more terms, this sum will cross the number 2. So, let us search out the value of the eleventh term, which is $\frac{1}{1024}$ = 0.0009765625.

Notice that the number of 0's after the decimal in this eleventh term is more than the number of 9's after the decimal in the sum of first ten terms. Therefore the sum of eleven terms of the series will not cross the number 2. Thus however, we can not find the exact sum of the series but it will go closer and closer towards the number 2 and will never cross it. Mathematician has chosen the name 'convergent' for this behavior of a series and so we say that the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\ldots$ converges to 2. More specifically, this number 2 is 'defined' to be the sum of the series. It should be noted that we are not saying that the exact sum of the series is 2.

Let us consider another infinite series

$$1 + 2 + 4 + 8 + 16 + \ldots \qquad \ldots(2)$$

Clearly, the sum of this series will go beyond all bounds and such series are named as divergent.

The main difference between the series (1) and (2) is that the terms of the series (1) are getting smaller whereas the terms of the series (2) are getting larger. What our intuition (feeling) says by noting the difference between two series. Those who goes so much by intuition will think that if the terms are getting smaller, then the series is convergent and of the term are getting larger then the series is divergent.

3.3 INFINITE SERIES

A series is the sum of the terms of a sequence. If $u_1, u_2, u_3, \ldots$ is a sequence then the sum $u_1 + u_2 + u_3 \ldots$ of all the terms is called an infinite series and is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

If $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is an infinite series, then

$$S_n = u_1 + u_2 + \ldots + u_n \; \forall \; n$$

The sequence $\{S_n\}$ is called the sequence of partial sums of the series and

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$

To every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

Note: An infinite series is said to converge, diverge or oscillate according as its sequence of partial sum $\{S_n\}$ converges, diverges or oscillates.

Remarks:

a. The series Σu_n is said to be convergent if the sequence $\{S_n\}$ of its partial sums is convergent. *i.e.*, Σu_n is convergent if $\lim_{n \to \infty} S_n$ = a finite quantity.

b. The series Σu_n diverges if the sequence $\{S_n\}$ of its partial sums diverges. *i.e.*, Σu_n is divergent if $\lim_{n \to \infty} S_n = +\infty$ or $-\infty$.

c. The series Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

d. The series Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

e. Convergence or divergence of an infinite series does not get affected if we add or omit some terms in series.

f. Nature of an infinite series does not change on multiplying the given series by a non zero constant.

g. Sum of two convergent series is always convergent.

SOME SOLVED EXAMPLES

Example 3.15. *Check the convergence of the series* $1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \ldots$

Solution. Here,

$$S_n = 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \ldots + (-1)^{n-1} \frac{1}{5^{n-1}}$$

$$= \frac{1-\left(-\frac{1}{5}\right)^n}{1-\left(-\frac{1}{5}\right)} = \frac{1}{1+\frac{1}{5}} = \frac{5}{6} \text{ as } n \to \infty \qquad \left[\therefore \text{ it is Finite G.P. use } \frac{a(1-r^n)}{1-r}\right.$$

$\therefore$ $\{S_n\}$ is a convergent sequence.

Therefore the given series is convergent.

Example 3.16. *Test the convergence of the series* $1^2 + 3^2 + 5^2 + 7^2 + \ldots$

Solution. For $1^2 + 3^2 + 5^2 + 7^2 + \ldots$

the n^{th} term $u_n = (2n-1)^2$

$$\lim_{n\to\infty} u_n = \infty$$

$\therefore$ Σu_n is not a convergent series and hence it is a divergent series.

Example 3.17. *Test the convergence of the series* $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$

Solution. The series can be written as

$$\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\ldots\right]$$

$$\therefore \qquad S_n = \frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

$$\Rightarrow \qquad S_n \to \frac{1}{2} \text{ as } n \to \infty$$

$\therefore$ $\sum_{n=1}^{\infty} u_n$ is convergent series.

Example 3.18. *Test the convergence of the series* $7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 + \ldots$

Solution. For the given series, the sequence $\{S_n\}$ of partial sum is given as

$$S_n = 7, \text{ if } n = 3m + 1, m \geq 0 \text{ a positive integer}$$
$$= 0, \text{ if } n = 3m, m \geq 0 \text{ a positive integer}$$
$$= 3, \text{ if } n = 3m - 1, m > 0 \text{ a positive integer.}$$

Hence $\{S_n\}$ is an oscillatory sequence.

Therefore $\sum_{n=1}^{\infty} u_n$ is an oscillatory series.

EXERCISE 3.4

1. Test the convergence of following series:

i. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} + \ldots \infty$

ii. $1^2 + 2^2 + 3^2 + \ldots n^2 + \ldots$

iii. $-1 - 2 - 3 - \ldots - n \ldots$

iv. $\sum_{n=1}^{\infty} (-1)^{n-1}$

v. $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots$ to ∞

2. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges to $\frac{3}{4}$.

3. Show that the series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1}$ converges to 4.

4. Show that the series $\sum_{n=1}^{\infty} n(-1)^n$ oscillates infinitely.

5. Test the convergence of the following series:
   a. $1 + 3 + 5 + 7 \ldots \infty$
   b. $-1 - 8 - 27 - 64 \ldots \infty$

Answers

1. i. Convergent ii. divergent to $+\infty$ iii. diverges to $-\infty$ iv. oscillates finitely
   v. Oscillates finitely 5. a. Divergent b. Divergent

3.3.1 Convergence or Divergence of Geometric Series

The geometric series $a + ar + ar^2 + \ldots + \infty$

i. converges if $|r| < 1$
ii. diverges if $r \geq 1$
iii. Oscillates finitely if $r = -1$
iv. Oscillates infinitely if $r < -1$

Proof: Here $S_n = a + ar + ar^2 + \ldots ar^{n-1}$

$$= \frac{a(1-r^n)}{1-r}, \text{ provided } r \neq 1$$

Case I. If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} \text{ which is unique and finite}$$

Thus the given series converges to $\frac{a}{1-r}$.

Case II. If $r > 1$, then $r^n \to \infty$ as $n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} = \infty$$

Thus the given series is divergent.

If $r = 1$, then $S_n = \underbrace{a + a + \ldots + a}_{n \text{ times}} = na$

$$\therefore \quad \lim_{n\to\infty} S_n = \infty$$

Thus the series is divergent.

Case III. If $r = -1$, then $S_n = a - a + a - a + \ldots$ upto n times

$$= \begin{cases} a \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$$

$\therefore$ Sequence $\{S_n\}$ oscillates and hence series $\sum_{n=1}^{\infty} a_n$ oscillates finitely.

Case IV. If $r < -1$

$\Rightarrow \quad -r > 1 \quad \Rightarrow \quad x > 1 \quad$ where $x = -r$

$\Rightarrow \quad x^n \to \infty$ as $\quad n \to \infty$

Now, $$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(1-(-x)^n)}{1+x}$$

$$= \begin{cases} \dfrac{a(1-x^n)}{1+x}, & \text{if } n \text{ is even} \\[2ex] \dfrac{a(1+x^n)}{1+x} & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} S_n = \begin{cases} -\infty, & \text{if } n \text{ is even} \\ \infty, & \text{if } n \text{ is odd} \end{cases}$$

Thus the sequence $\{S_n\}$ oscillates infinitely and hence the given series oscillates infinitely.

Examples of Geometric Series (Convergence or Divergence)

1. $$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

   It is geometric series with $r = \dfrac{1}{2}$

   $$r = \frac{1}{2} < 1$$

   $\therefore$ it is convergent series

2. $$\sum_{n=1}^{\infty} 3^n = 3 + 3^2 + 3^3 + \ldots$$

   It is geometric series with $r = 3$

   $$r = 3 > 1$$

   $\therefore$ it is divergent.

3. $$\frac{3}{4} - \frac{3}{4} + \frac{3}{4} - \frac{3}{4} + \ldots$$

   It is geometric series with $r = -1$

   $$r = -1$$

   $\therefore$ The given series oscillates finitely.

3.3.2 Series of Positive Terms

A series $\sum_{n=1}^{\infty} a_n$ for which $a_n > 0$ for all n, is called a series of positive terms.

*__Basic Result:__ A necessary condition for convergence: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$, and the converse is not always true.

Results:

a. (A Fundamental test for a positive term series) A positive term series Σa_n is convergent if and only if its sequence $<S_n>$ of partial sums is bounded above.
i.e., A positive term series Σa_n converges.
$\Rightarrow$ $S_n < k \; \forall \; n$ (k being some positive real number).

b. A monotonic sequence either converges or diverges but cannot oscillate.
Hence a positive term series either converges or diverges.
i.e., A positive term series cannot be oscillating.

Note:

a. If $\lim_{n \to \infty} a_n \neq 0$ then the positive terms series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$

b. If $\lim_{n \to \infty} a_n = 0$, then there is not conclusion about the behaviour of positive terms series $\sum_{n=1}^{\infty} a_n$

3.3.3 Cauchy's General Principle of Convergence

A necessary and sufficient condition for series $\sum_{n=1}^{\infty} a_n$ to converge is that for each $\varepsilon > 0$, there exists a positive integer m such that

$$| a_{m+1} + a_{m+2} + ... + a_n | < \varepsilon \text{ for all } n \geq m.$$

SOME SOLVED EXAMPLES

Example 3.19. *Show that the series* $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$ *does not converge.*

Solution. Suppose the given series converges. Then the sequence $<S_n>$ of partial sums of the given series is convergent. By Cauchy's principle of convergence for sequence, for $\varepsilon = \frac{1}{2}$ there exist a positive integer m such that

$$| S_n - S_m | < \frac{1}{2} \; \forall \; n \geq m$$

$$\left|\left(1+\frac{1}{2}+\ldots+\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}\right)-\left(1+\frac{1}{2}+\ldots\frac{1}{m}\right)\right| < \frac{1}{2}\ \forall\ n \geq m$$

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}<\frac{1}{2}\ \forall\ n \geq m \qquad \ldots(i)$$

On taking $n = 2m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n} = \frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{2m} > \frac{1}{2m}+\frac{1}{2m}+\ldots+\frac{1}{2m}$$

$$= m\ .\ \frac{1}{2m} = \frac{1}{2}$$

$$\therefore \quad \frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n} > \frac{1}{2}\ (n = 2m > m) \qquad \ldots(ii)$$

Since (i) and (ii) are contradictory statement, the given series does not converge.

3.3.4 Comparison Test

Test I

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms and $\sum_{n=1}^{\infty} b_n$ is convergent and there is a positive constant κ such that $a_n \leq \kappa\ b_n$, $\forall\ n > m$ then $\sum_{n=1}^{\infty} a_n$ is also convergent.

Test II

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two positive terms series and $\sum_{n=1}^{\infty} b_n$ is divergent and there is a positive constant κ such that $a_n > \kappa\ b_n$, $\forall\ n > m$ then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Test III

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two positive terms series and κ, K are two positive numbers such that $\kappa\ b_n < a_n < K\ .\ b_n$ for all n, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges or diverges together.

Test IV

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series or positive terms such that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$ (Finite and non zero), then both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

Remark: In the above test the condition that $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}$ is finite and non zero cannot be dropped.

Consider $$a_n = \frac{1}{n} \text{ and } b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n = \infty$$

Here $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ do not converge or diverge together as $\sum\limits_{n=1}^{\infty} a_n$ is divergent and $\sum\limits_{n=1}^{\infty} b_n$ is convergent.

Test V

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are two series of positive terms such that

a. $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = 0$ and $\sum\limits_{n=1}^{\infty} b_n$ is convergent, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

b. $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \infty$ and $\sum\limits_{n=1}^{\infty} b_n$ is divergent, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Test VI

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are two series of positive terms such that $\dfrac{a_n}{a_{n+1}} \geq \dfrac{b_n}{b_{n+1}}$ for all $n \geq m$, where m is some positive integer, then

i. $\sum\limits_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ converges

ii. $\sum\limits_{n=1}^{\infty} a_n$ diverges $\Rightarrow$ $\sum\limits_{n=1}^{\infty} b_n$ diverges.

3.3.5 Important Test for Comparison

[$\dfrac{1}{n^p}$ **Hyper harmonic series or *p*-series**]

Theorem: The series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^p} = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} \ldots + \dfrac{1}{n^p} + \ldots$

i. converges if $p > 1$
ii. diverges if $p \leq 1$

Proof: Case I. If $p > 1$

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2}$$

....

....

....

$$\frac{1}{(2^n)^p} + \ldots + \frac{1}{(2^{n+1}-1)^p} < \frac{1}{(2^{p-1})^n}$$

Adding the above inequalities, we have

$$S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \ldots + \frac{1}{(2^{p-1})^n}$$

In the above inequality the right hand side is a geometrical series with common ratio $\frac{1}{2^{p-1}} < 1$

$$S_{2^{n+1}-1} < \frac{1\left[1 - \frac{1}{(2^{p-1})^n}\right]}{1 - \frac{1}{2^{p-1}}} \qquad \left[\because \; S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$S_{2^{n+1}-1} < \frac{1}{1 - \frac{1}{2^{p-1}}}$$

or $\quad S_{2^{n+1}-1} < k$ where $k = \dfrac{1}{1 - \frac{1}{2^{p-1}}}$ is fixed

But $\quad S_n < S_{2^{n+1}-1}$ for all n

$S_n < k$ for all n

$\Rightarrow$ $<S_n>$ is bounded above sequence. But every positive terms sequence which is bounded above is convergent.

$\Rightarrow$ $<S_n>$ is convergent and hence the given series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Case II. If $p = 1$ in this case we have the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

$$1 + \frac{1}{2} > \frac{1}{2}$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{2}{4}=\frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

we have

$$1+\frac{1}{2} > \frac{1}{2}$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{2}$$

....

....

$$\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n} > \frac{1}{2}$$

Adding the above inequality, we have

$$S_{2^n} > \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots \text{ upto } n \text{ terms}$$

$$S_{2^n} > \frac{n}{2}$$

$\Rightarrow$ Sequence $<S_{2^n}>$ is not bounded above as n can take sufficient large values.

Hence the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent to $+\infty$.

Case III. If $p < 1$

Here $n^p < n$ for all n or $\frac{1}{n^p} > \frac{1}{n}$ for all n.

By case II, $\sum_{n=1}^{\infty}\frac{1}{n}$ is also divergent.

$\therefore$ By comparison test $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is also divergent.

This proves the result.

SOME SOLVED EXAMPLES

Example 3.20. *Test the convergence of the series:*

i. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\infty$

ii. $\frac{1}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{3}{5\cdot 7}+\ldots\infty$

iii. $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots\infty$

Solution. i. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\infty$

This series can be written as

$$\Sigma a_n = \sum \frac{1}{n(n+1)}$$

$$a_n = \frac{1}{n^2\left(1+\frac{1}{n}\right)}$$

Let $$b_n = \frac{1}{n^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2\left(1+\frac{1}{n}\right)}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 \text{ non zero and finite}$$

By comparison test, series Σa_n and Σb_n converges or diverges together.

But $$\Sigma b_n = \sum\frac{1}{n^2} \text{ is of the form } \sum\frac{1}{n^p}$$

Here $$p = 2 > 1$$

$\therefore$ Σb_n is convergent.

$\Rightarrow$ Σa_n is convergent.

ii. **Hint:** $a_n = n/(2n-1)\,(2n+1)$ and $b_n = 1/n$

Answer. Given series is divergent.

iii. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+\ldots\infty$

Here $$a_n = \frac{1}{\sqrt{n(n+1)}} = \frac{1}{n\sqrt{1+\frac{1}{n}}}$$

Let $$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

$\therefore$ Σa_n and Σb_n both converges or diverge together.

But $\Sigma b_n = \sum \frac{1}{n}$ is divergent (since $p = 1$)

Hence given series is divergent.

Example 3.21. *Check the convergence of the series* $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \ldots \infty$

Solution. Hint: $a_n = (n+1)/n^p$ and $b_n = 1/n^{(p-1)}$

Answer: Given series is convergent for $p > 2$ and divergent for $p \le 2$.

Example 3.22. *Check the convergence of the series* $\sum_{n=1}^{\infty} \frac{n+1}{n(2n-1)}$.

Answer. Here $$a_n = \frac{n+1}{n(2n-1)} = \frac{n\left(1+\frac{1}{n}\right)}{n^2\left(2-\frac{1}{n}\right)}$$

Let $$b_n = \frac{1}{n}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2} \text{ (finite and non-zero)}$$

By comparison test Σa_n and Σb_n converges or diverges together.

But $$\Sigma b_n = \frac{1}{n} \text{ is divergent} \qquad (\because \ p = 1, \text{by } p\text{-test})$$

So given series is divergent.

Example 3.23. *Test the convergence of the following series*

i. $\sum_{n=1}^{\infty} \frac{1}{n^p\,(n+1)^p}$

ii. $\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \ldots \infty$

iii. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

iv. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+1}$

Solution. i. **Hint:** $a_n = 1/(n^p)\,(n+1)^p$ and $b_n = 1/n^{2p}$

Answer: Given series is convergent if $p > 1/2$ and divergent if $p \le 1/2$

ii. **Hint:** $a_n = (n+1)^p/n^q$ and $b_n = 1/n^{(q-p)}$

Answer: Given series is convergent if $q > p + 1$ and divergent if $q \le p + 1$

iii. **Hint:** $a_n = \sqrt{n}\,/(n^2+1)$ and $b_n = 1/n^{(3/2)}$

Answer: Given series is convergent.

iv. **Hint:** $a_n = \sqrt{(n^2-1)}\,/(n^3+1)$ and $b_n = 1/n^2$

Answer: Given series is convergent

Example 3.24. *Check the convergence of the given series:*

a. $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\ldots\infty$ b. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots\infty$

Solution. a. Hint: $a_n = \sqrt{n}/(2n+3)$ and $b_n = 1/n^{(1/2)}$

Answer. Given series is divergent.

b. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots\infty$

Here $$a_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

and let $$b_n = \frac{1}{n^3}$$

now $$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1 \text{ (finite and non-zero)}$$

Σa_n and Σb_n converge or diverge together.

But $\Sigma b_n = \frac{1}{n^3}$ is convergent since $p = 3 > 1$, by p-test

Hence given series is convergent.

Example 3.25. *Check the convergence of the series* $\sum\left(\sqrt{n^2+1}-n\right)$

Solution. Here
$$a_n = \sum\left(\sqrt{n^2+1}-n\right)$$
$$= \sqrt{n^2+1}-n\times\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$
$$= \frac{1}{\sqrt{n^2+1}+n} = \frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)}$$

Let $$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\left(\sqrt{1+\frac{1}{n^2}}+1\right)} = \frac{1}{2}$$

By comparison test Σa_n and Σb_n converge or diverge together.

But $\Sigma b_n = \frac{1}{n}$ is divergent (since $p = 1$ by p test)

Hence given series is divergent.

Example 3.26. *Check the convergence of the series:*

$$\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$$

Solution. Try yourself.

Example 3.27. *Test the convergence of the series* $\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{2}+\frac{\sqrt{4}-\sqrt{3}}{3}+\ldots$

Solution. Hint: $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$ and $b_n = 1/n^{(3/2)}$

Answer: Given series is convergent.

Example 3.28. *Test the convergence of the series* $\sum\left\{\sqrt[3]{n+1}-\sqrt[3]{n}\right\}$

Solution. Answer: Given series is divergent.

Example 3.29. *Check the convergence of the following series:*

a. $\sum\frac{1}{\sqrt{n}}\sin\frac{1}{n}$

b. $\Sigma \cot^{-1} n^2$

c. $\sum\left(\frac{1}{n}-\log\frac{n+1}{n}\right)$

d. $\sum_{n=1}^{\infty}\left(1-\cos\frac{\pi}{n}\right)$

Solution. a. **Hint:** Use $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}$ ......

Answer: Given series is convergent.

b.
$$\Sigma a_n = \Sigma \cot^{-1} n^2$$

$$a_n = \cot^{-1} n^2$$

$$= \tan^{-1}\frac{1}{n^2} \qquad \left[\because \tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right]$$

$$= \frac{1}{n^2}-\frac{1}{3}\left(\frac{1}{n^2}\right)^3+\frac{1}{5}\left(\frac{1}{n^2}\right)^5-\frac{1}{7}\left(\frac{1}{n^2}\right)^7+\ldots$$

Let
$$b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = 1$$

By comparison test both a_n and b_n converge or diverge together.

But $b_n = \frac{1}{n^2}$ is convergent as $p = 2 > 1$ by p-test

$\Rightarrow \quad \Sigma a_n$ is convergent.

c.
$$a_n = \frac{1}{n}-\log\frac{n+1}{n}$$

$$= \frac{1}{n}-\log\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} \ldots\right)$$

$$= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} \ldots$$

$$= \frac{1}{n^2}\left[\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} \ldots\right]$$

Let $$b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2} \text{ (finite and non-zero)}$$

Σa_n and Σb_n both converge or diverge together.

But $\Sigma b_n = \dfrac{1}{n^2}$ is convergent as $p = 2 > 1$

so a_n is also convergent.

d. **Hint:** use $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$ ......

Answer: Series is convergent.

Example 3.30. *Check the convergence of the series* $\sum \dfrac{x^{n-1}}{1+x^n}$, $x > 0$.

Solution. Let, $$a_n = \frac{x^{n-1}}{1+x^n}$$

To check the convergence, we have the following cases:

Case I. $0 < x < 1$

Then $x^n \to 0$ as $n \to \infty$

Let $b_n = x^{n-1}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+x^n} = \frac{1}{1+0} = 1 \qquad \text{(which is finite and non-zero)}$$

Σa_n and Σb_n both converge or diverge together.

But $\Sigma b_n = \Sigma x^{n-1}$ is a G.P. series with common ratio $x < 1$

$\therefore$ Σb_n is convergent.

$\therefore$ Σa_n is convergent.

Case II. If $x = 1$

$$\lim_{n\to\infty} a_n = \frac{1}{1+1} = \frac{1}{2} \text{ (non-zero and finite)}$$

If $\lim\limits_{n\to\infty} a_n \neq 0$, then Σa_n does not change.

Σa_n is a divergent series.

Case III. If $x > 1$, then $0 < \frac{1}{x} < 1$ and

$$\frac{1}{x^n} \to 0 \text{ as } n \to \infty$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{x^n \cdot x\left(1+\frac{1}{x^n}\right)} = \lim_{n\to\infty} \frac{1}{x\left(1+\frac{1}{x^n}\right)} = \frac{1}{x} \neq 0$$

Hence series is divergent.

Thus the given series is convergent if $x < 1$, divergent if $x \geq 1$.

EXERCISE 3.5

Test the convergence or divergence of the following series:

1. $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\ldots\infty$

2. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots\infty$

3. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\ldots\infty$

4. $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots\infty$

5. $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\infty$

6. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$

7. $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots$

8. $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots+\sqrt{\frac{n}{2(n+1)}}+\ldots$

9. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$

10. $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots$

11. $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\ldots$

12. $\sum\frac{2n^3+5}{4n^5+1}$

13. $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

14. $\sum\frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$

15. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+\ldots$

16. $\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\ldots$

17. $\frac{1}{a(a+b)}+\frac{a}{(a+2b)(a+3b)}+\frac{1}{(a+4b)(a+5b)}+\ldots\infty, a>0, b>0$

18. $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n^p}$

19. $\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{1}{n}$

20. $\sum_{n=1}^{\infty}\frac{1}{x^n+x^{-n}},\ x>0$

Answers

1. Convergent	**2.** Convergent	**3.** Divergent	**4.** Convergent
5. Convergent	**6.** Convergent	**7.** Divergent	**8.** Divergent
9. Divergent	**10.** Convergent	**11.** Convergent for $p > 1$, divergent for $p \le 1$	
12. Convergent	**13.** Convergent	**14.** Divergent	**15.** Divergent
16. Divergent	**17.** Convergent	**18.** Convergent if $p > \frac{1}{2}$ and divergent if $p \le \frac{1}{2}$	
19. Convergent	**20.** Converges for $x < 1$ and $x > 1$ but diverges for $x = 1$		

3.3.6 D' Alembert's Ratio Test

If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms such that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = l$, then

i. For, $l > 1$, the series is convergent.
ii. For $l < 1$, the series is divergent.
iii. For $l = 1$, no conclusion (*i.e.*, the test fails).

SOME SOLVED EXAMPLES

Example 3.31. *Discuss the convergence of the following series:*

i. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\ldots\infty$

ii. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots\infty$

iii. $\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots\infty$

iv. $\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\ldots\infty$

v. $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\left(\frac{1.2.3.4}{3.5.7.9}\right)^2\ldots\infty$

Solution. i. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\ldots\infty$

Here $$a_n = \frac{n!}{n^n},\ a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{n!}{n^n}\times\frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{n!(n+1)^n\,.\,(n+1)}{n^n(n+1)\,n!}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}$$

$$= \lim_{n \to \infty} \frac{n^n \left(1+\frac{1}{n}\right)^n}{n^n}$$

$$= \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$\therefore$ Σa_n is convergent.

ii. **Hint:** $a_n = n/1 + 2^n$ **Answer:** convergent

iii. **Hint:** $a_n = \dfrac{2.5.8.11 \ldots (3n-1)}{1.5.9.13 \ldots [4(n-1)+1]}$ **Solution:** Try yourself.

iv. **Hint:** $a_n = (n + 1)!/3^n$ **Answer:** divergent

v. $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \left(\frac{1.2.3.4}{3.5.7.9}\right)^2 \ldots \infty$

Solution. Here

$$a_n = \left[\frac{1.2.3.4 \ldots n}{3.5.7.9. \ldots (2n+1)}\right]^2$$

$$a_{n+1} = \left[\frac{1.2.3.4 \ldots n(n+1)}{3.5.7.9. \ldots (2n+1)(2n+3)}\right]^2$$

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \left(\frac{(2n+3)}{(n+1)}\right)^2 = \lim_{n \to \infty} \frac{n^2}{n^2} \left(\frac{\left(2+\frac{3}{n}\right)}{1+\frac{1}{n}}\right)^2$$

$$= 4 > 1$$

$\therefore$ Σa_n is convergent.

Example 3.32. *Discuss the convergence of the following series:*

i. $\sum \dfrac{2^{n-1}}{3^n + 1}$

ii. $\sum \dfrac{1}{n!}$

iii. $\sum \dfrac{x^n}{3^n \cdot n^2}, x > 0$

iv. $\sum \dfrac{x^n}{n}, x > 0$

v. $\sum \dfrac{n}{n^2 + 1} \cdot x^n, x > 0$

vi. $\sum \left(\dfrac{n^2}{2^n} + \dfrac{1}{n^2}\right)$

Solution. i. $\sum \dfrac{2^{n-1}}{3^n + 1}$

Here

$$a_n = \sum \frac{2^{n-1}}{3^n + 1}$$

and

$$a_{n+1} = \frac{2^n}{3^{n+1} + 1}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{2^{n-1}}{3^n+1}\times\frac{3^{n+1}+1}{2^n}$$

$$= \lim_{n\to\infty}\frac{2^n\,.\,3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)}{2\,.\,3^n\left(1+\frac{1}{3^n}\right)2^n}$$

$$= \lim_{n\to\infty}\frac{3}{2}\left(\frac{1+\frac{1}{3^{n+1}}}{1+\frac{1}{3^n}}\right) = \frac{3}{2} > 1$$

$\therefore$ Σa_n is convergent.

ii. Try yourself.

Answer: Convergent

iii. Try yourself

Answer: series is convergent for $3 \geq x$ and divergent for $3 < x$.

iv. Try yourself

Answer. series is convergent if $x < 1$ and divergent if $x \geq 1$.

v. $\sum \frac{n}{n^2+1}x^n, x > 0$

Here $a_n = \frac{nx^n}{n^2+1}$ $\quad a_{n+1} = \frac{(n+1)\,.\,x^{n+1}}{(n+1)^2+1}$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{n}{n^2+1}x^n\times\frac{(n+1)^2+1}{(n+1)x^{n+1}}$$

$$= \lim_{n\to\infty}\frac{n\,.\,x^n\,.\,n^2\left[\left(1+\frac{1}{n}\right)+\frac{1}{n^2}\right]}{n\,.\,n^2\left(1+\frac{1}{n^2}\right)\left(1+\frac{1}{n}\right)x^n\,.\,x} = \frac{1}{x}$$

By ratio test Σa_n is convergent if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and divergent if $x > 1$, test fails if $x = 1$.

Put $x = 1$ in given series then

$$a_n = \frac{n}{n^2+1} = \frac{1}{n\left(1+\frac{1}{n^2}\right)}$$

Let $b_n = \frac{1}{n}$, which is divergent.

Hence series is convergent if $x < 1$ and divergent for $x \geq 1$.

vi. $\sum\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$

$$\sum\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right) = \sum a_n + \sum b_n$$

Here $$a_n = \frac{n^2}{2^n},\ a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$\therefore$ $$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{n^2}{2^n}\times\frac{2^{n+1}}{(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{n^2 \,.\, 2^n \,.\, 2}{2^n \,.\, n^2\left(1+\frac{1}{n^2}\right)^2}$$

$$= \lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n^2}\right)^2} = 2 > 1$$

Hence $\sum\frac{n^2}{2^n}$ is convergent.

Also $$\Sigma b_n = \sum\frac{1}{n^2} \text{ is convergent by } p\text{-test.}$$

Hence given series is convergent.

Example 3.33. *Discuss the convergence of the series* $\sum\sqrt{\frac{n+1}{n^3+1}}\,.\,x^n,\ x > 0.$

Solution. Here $$a_n = \sum\sqrt{\frac{n+1}{n^3+1}}\,.\,x^n$$

$$a_{n+1} = \sum\sqrt{\frac{n+2}{(n+1)^3+1}}\,.\,x^{n+1}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n^3+1}}\,.\,x^n \times \sqrt{\frac{(n+1)^3+1}{(n+2)}}\,.\,\frac{1}{x^{n+1}} = \frac{1}{x}$$

Σa_n is convergent if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and divergent if $\frac{1}{x} < 1$, *i.e.*, $x > 1$.

Test fails if $$\frac{1}{x} = 1 \quad\Rightarrow\quad x = 1,$$

Put $x = 1$ in given series $a_n = \sqrt{\frac{n+1}{n^3+1}} = \frac{1}{n}\sqrt{\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n^3}\right)}}$

Let $\Sigma b_n = \sum \frac{1}{n}$ which is divergent by p-test

Hence given series is convergent, for $x < 1$ and divergent for $x \geq 1$.

Example 3.34. $x + 2x^2 + 3x^3 + 4x^4 + ..., \infty$

Solution. Here

$$a_n = nx^n$$

$$a_{n+1} = (n+1)x^{n+1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{nx^n}{(n+1)\,x^{n+1}} = \frac{1}{x}$$

by D's Alembert's ratio test, this series is convergent if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and divergent.

If $\frac{1}{x} < 1$ *i.e.,* $x > 1$, test fails if $\frac{1}{x} = 1$ *i.e.,* $x = 1$

Put $x = 1$ in given series

$$\sum a_n = \sum n\,,\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$$

$\therefore$ Series is divergent for $x = 1$

Hence given series is convergent for $x < 1$, divergent for $x \geq 1$.

Example 3.35. $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + ... \frac{n^2-1}{n^2+1}x^n$, *Check the convergence of the series.*

Solution. Let

$$a_n = \frac{n^2-1}{n^2+1}x^n,\ a_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1}\,.\,x^{n+1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^2-1}{n^2+1}\,.\,x^n \times \frac{(n+1)^2+1}{(n+1)^2-1}\,.\,\frac{1}{x^{n+1}}$$

$$= \frac{1}{x}$$

Σa_n is convergent if $\frac{1}{x} > 1$, *i.e.,* $x < 1$, divergent if $\frac{1}{x} < 1$ *i.e.,* $x > 1$ test fails if $x = 1$.

Put $x = 1$ in given series, we get

$$a_n = \frac{n^2-1}{n^2+1},\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$$

By the result for positive term series $\sum a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is definitely divergent.

$\therefore$ Σa_n is divergent for $x = 1$

$\sum \frac{n^2-1}{n^2+1}\,.\,x^n$ is convergent for $x < 1$, divergent for $x \geq 1$.

Example 3.36. *Test the convergence of* $\sum \frac{3^n-2}{3^n+1}\,.\,x^{n-1}$*, $x > 0$.*

Solution. Try yourself.

Answer: Convergent for $x < 1$ and divergent for $x \geq 1$

Example 3.37. *Test the convergence of* $\sum_{n=1}^{\infty} \frac{n \,.\, x^n}{(n+1)(n+2)}$

Solution. Try yourself.

Answer: Convergent for $x < 1$ and divergent for $x \geq 1$

Example 3.38. $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots \infty$

Solution. Try yourself.

Answer: Convergent for $x^2 < 1$ and divergent for $x^2 \geq 1$

Example 3.39. $\frac{4}{18} + \frac{4 \,.\, 12}{18 \,.\, 27} + \frac{4 \,.\, 12 \,.\, 20}{18 \,.\, 27 \,.\, 36} + \infty$

Solution. Hint: Series is convergent.

Answer: $\frac{4.12.20 \ldots (8n-4)}{18.27.36 \ldots (9n+9)}$

EXERCISE 3.6

Discuss the convergence of the following series:

1. $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots, (p > 0)$

2. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} \ldots + \frac{1}{2^{n-1}+1} + \ldots$

3. $\frac{1^2 \,.\, 2^2}{1!} + \frac{2^2 \,.\, 3^2}{2!} + \frac{3^2 \,.\, 4^2}{3!} + \frac{4^2 \,.\, 5^2}{4!} + \ldots\ldots \infty$

4. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots \infty$

5. $\sum \frac{n^3 + a}{2^n + a}$

6. $\sum \frac{n!\, 3^n}{n^n}$

7. $\sum \frac{n!}{n^n}$

8. $\sum \frac{n^3 - n + 1}{n!}$

9. $\frac{1}{1\cdot 2\cdot 3} + \frac{x}{4\cdot 5\cdot 6} + \frac{x^2}{7\cdot 8\cdot 9} + \ldots \infty, (x > 0)$

10. $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots + \frac{x^n}{n^2+1} + \ldots \infty$

11. $\frac{x}{1\cdot 3} + \frac{x^2}{3\cdot 5} + \frac{x^3}{5\cdot 7} + \ldots$

12. $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^2}{4\sqrt{5}} + \ldots \infty$

13. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

14. $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

15. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \frac{x^3}{3+\sqrt{3}} + \ldots \infty$

16. $\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)^2 \,.\, 2^n}$

17. $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots\infty$

18. $\sum_{n=1}^{\infty}\frac{x^{n+1}}{(n+1)\sqrt{n}}$

19. $1+\frac{1^2\cdot 2^2}{1\cdot 3\cdot 5}+\frac{1^2\cdot 2^2\cdot 3^2}{1\cdot 3\cdot 5\cdot 7\cdot 9}+\infty$

20. $\sum_{n=1}^{\infty}\frac{3\cdot 6\cdot 9\ldots 3n}{4\cdot 7\cdot 10\ldots(3n+1)}\cdot\frac{5^n}{3n+2}$

21. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n,(x>0)$

22. $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots$

23. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots\frac{2^{n+1}-2}{2^{n+1}+1}.x^n+\ldots(x>0)$

24. $1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3a+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots$

Answers

1. Convergent
2. Convergent
3. Divergent
4. Convergent
5. Convergent
6. Divergent
7. Convergent
8. Convergent
9. Convergent if $x\le 1$, divergent if $x>1$
10. Convergent if $x\le 1$, divergent for $x>1$
11. Convergent for $x\le 1$, divergent if $x>1$
12. Convergent if $x\le 1$, divergent if $x>1$
13. Convergent
14. Convergent
15. Convergent if $x<1$, divergent if $x\ge 1$
16. Convergent if $x\le 2$, divergent if $x>2$
17. Convergent
18. Convergent if $x\le 1$, divergent if $x>1$
19. Convergent
20. Divergent
21. Convergent if $x<1$, divergent if $x\ge 1$
22. Convergent if $x^2\le 1$, divergent if $x^2>1$
23. Convergent if $x<1$, divergent if $x\ge 1$
24. Convergent if $\beta>\alpha>0$, divergent if $\alpha\ge\beta>0$

3.3.7 Cauchy's Root Test

If Σa_n is a positive term series and $\lim_{n\to\infty}(a_n)^{1/n}=l$, then

i. Σa_n is convergent if $l<1$

ii. Σa_n is divergent if $l>1$

If $l=1$, the test fails *i.e.*, no conclusion can be drawn about the convergence or divergence of the series.

Note: This test is useful when $<a_n>$ involves expression with nth power.
If this test fails, comparison test may be used.

SOME SOLVED EXAMPLES

Example 3.40. *Check the convergence of* $\sum \frac{1}{n^n}$.

Solution. Here

$$\Sigma a_n = \sum \frac{1}{n^n}$$

$$a_n = \frac{1}{n^n}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

By Cauchy's test, this series is convergent.

Example 3.41. *Test the convergence of* $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

Solution. Here

$$a_n = \frac{1}{(\log n)^n}$$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\log n} < 1$$

$\therefore$ series is convergent.

Example 3.42. *Test the convergence of* $\sum \left(\frac{n+1}{3n}\right)^n$

Solution. Try yourself.
Answer: Convergent.

Example 3.43. *Test the convergence of* $\sum \frac{(n - \log n)^n}{2^n . n^n}$

Solution. Try yourself.
Answer: Convergent.

Example 3.44. *Check the convergence of* $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Solution. Here

$$a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n\sqrt{n}}}\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e}$$

which is less than 1 $\left(\because \frac{1}{e} < 1\right)$

$\therefore$ Series is convergent.

Example 3.45. *Test the convergence of* $\sum \left(\frac{nx}{(n+1)}\right)^n$.

Solution. Here $$a_n = \left(\frac{nx}{(n+1)}\right)^n$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{nx}{n\left(1+\frac{1}{n}\right)}$$

$$= x$$

Series is convergent if $x < 1$,

divergent if $x > 1$

and test fails if $x = 1$

Put $x = 1$ in given series

$$a_n = \left(\frac{n}{(n+1)}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} = \lim_{n\to\infty} \left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{1}{e} \neq 0$$

($\because$ if $\lim\limits_{n\to\infty} a_n \neq 0$, then Σa_n is not convergent)

Hence Σa_n is divergent.

Given series is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 3.46. *Test the convergence of* $\sum 5^{-n-(-1)^n}$

Solution. Try yourself.

Answer: Convergent.

Example 3.47. *Check the convergence of* $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots\ldots \infty$

Solution. Here $$a_n = \frac{x^{n-1}}{n^{n-1}} \text{ (neglecting Ist term)}$$

$$(a_n)^{1/n} = \frac{x^{\frac{n-1}{n}}}{n^{\frac{n-1}{n}}} = \frac{x^{1-\frac{1}{n}}}{n^{1-\frac{1}{n}}}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{x \,.\, x^{\frac{-1}{n}}}{n \,.\, n^{\frac{-1}{n}}}$$

$$\lim_{n\to\infty} \frac{x}{n} = 0 < 1$$

$\therefore$ Series is convergent.

Example 3.48. *Test the convergence of* $\sum \frac{(1+nx)^n}{n^n}$.

Solution. Try yourself.

Answer: Convergent for $x < 1$ and divergent for $x \geq 1$

EXERCISE 3.7

Test the convergence or divergence of the following series:

1. $\sum \left(\frac{n}{(n+1)}\right)^{n^2}$ or $\sum \left(1+\frac{1}{n}\right)^{-n^2}$

2. $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$

3. $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots (x>0)$

4. $\sum_{n=1}^{\infty} \frac{(n+1)^n \,.\, x^n}{n^{n+1}}$

5. $\frac{2x}{1^2}+\frac{3^2 x^2}{2^3}+\frac{4^3 x^3}{3^4}+\ldots+\frac{(n+1)^n \,.\, x^n}{n^{n+1}}+\ldots$

Answers

1. Convergent **2.** Convergent **3.** Convergent if $x < 1$, divergent if $x \geq 1$

4. Convergent if $x < 1$, divergent if $x \geq 1$ **5.** Convergent if $x < 1$, divergent if $x \geq 1$

3.3.8 Raabe's Test

If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms such that $\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] = l$

Then the series is

i. Convergent if $l > 1$

ii. Divergent if $l < 1$

iii. Test fails if $l = 1$

Remarks:

a. Raabe's test is used when D' Alembert Ratio test fails and when $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ does not involve the number *e*.

b. Logarithmic test is used when Ratio test fails and $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}}$ contains e.

c. Raabe's test is inconclusive when $\lim\limits_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = 1$

3.3.9 Logarithmic Test

If the series $\sum_{n=1}^{\infty} a_n$ is positive term series such that $\lim\limits_{n\to\infty} n \log \frac{a_n}{a_{n+1}} = l$, then the series is

i. convergent if $l > 1$
ii. divergent if $l < 1$
iii. test fails if $l = 1$

3.3.10 Gauss Test

If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms such that $\frac{a_n}{a_{n+1}}$ can be expanded in the form

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

Then Σa_n converges if $\lambda > 1$ and diverges if $\lambda \leq 1$.

Remarks:

a. For application of Gauss test, expand $\frac{a_n}{a_{n+1}}$ in power of $\frac{1}{n}$ as $\frac{a_n}{a_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$, where $O\left(\frac{1}{n^2}\right)$ stands for the terms of higher power of $\frac{1}{n}$.

b. The test never fails as we know that the series diverges for $\lambda = 1$. Moreover the test is applied after the failure of Ratio test and when it is possible to expand $\frac{a_n}{a_{n+1}}$ in power of $\frac{1}{n}$ by binomial theorem or by any other method.

SOME SOLVED EXAMPLES

Example 3.49. *Discuss the convergence of the following series:*

a. $1 + \frac{2^2}{3^2} + \frac{2^2 . 4^2}{3^2 . 5^2} + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + \ldots \infty$

b. $1 + \frac{3}{7}x + \frac{3 \cdot 6}{7 \cdot 10}x^2 + \frac{3 \cdot 6 \cdot 9}{7 \cdot 10 \cdot 13}x^3 + \ldots \infty$

c. $1 + \frac{2}{1} . \frac{1}{2} + \frac{2 \cdot 4}{1 \cdot 3} . \frac{1}{3} + \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5} . \frac{1}{4} + \ldots\ldots \infty$

d. $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots \infty$

Solution. a. We have, $1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\text{......}\infty$

Neglecting 1st term, $a_n = \frac{2^2\cdot 4^2\cdot 6^2 \ldots (2n)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2}$

$$a_{n+1} = \frac{2^2\cdot 4^2\cdot 6^2 \ldots (2n)^2\,(2n+2)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2\,(2n+3)^2}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(2n+3)^2}{(2n+2)^2}=1$$

$\therefore$ D's Alembert's ratio test fail, Now we apply Raabe's test.

$$\frac{a_n}{a_{n+1}}-1 = \frac{(2n+3)^2}{(2n+2)^2}-1$$

$$= \frac{4n^2+9+12n-4n^2-4-8n}{(2n+2)^2}=\frac{4n+5}{(2n+2)^2}$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] = \lim_{n\to\infty}\frac{4+\frac{5}{n}}{\left(2+\frac{2}{n}\right)^2}=1$$

$\therefore$ Raabe's test fails.

Now by Gauss test

$$\frac{a_n}{a_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2}$$

$$= \left(1+\frac{3}{2n}\right)^2\left(1+\frac{2}{2n}\right)^{-2}$$

$$= \left(1+\frac{3}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)$$

$$= 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing with $1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right) \quad\Rightarrow\quad \lambda = 1$

$\therefore$ By Gauss test series is divergent.

b. $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots\infty$

Neglecting 1st term, $a_n = \frac{3\cdot 6\cdot 9\ldots 3n}{7\cdot 10\cdot 13\ldots(3n+4)}x^n$

$$a_{n+1} = \frac{3\cdot 6\cdot 9\ldots(3n)\,(3n+3)}{7\cdot 10\cdot 13\ldots(3n+4)\,(3n+7)}x^{n+1}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{3n+7}{3n+3}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

Series is convergent if $\frac{1}{x} > 1$ *i.e.*, $x < 1$

divergent if $\frac{1}{x} < 1$ *i.e.*, $x > 1$

and test fails if $x = 1$

Now apply Raabe's test, put $x = 1$, we get

$$\frac{a_n}{a_{n+1}} - 1 = \frac{3n+7}{3n+3} - 1 = \frac{3n+7-3n-3}{3n+3}$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \lim_{n\to\infty}\frac{4n}{3n+3} = \frac{4}{3} > 1 \quad \therefore \quad \Sigma a_n \text{ is convergent.}$$

Hence given series is convergent for $x \le 1$

and divergent for $x > 1$

c. $1 + \frac{2}{1}\cdot\frac{1}{2} + \frac{2\cdot 4}{1\cdot 3}\frac{1}{3} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\cdot\frac{1}{4} + \ldots\infty$

Neglecting first term, $\quad a_n = \frac{2\cdot 4\cdot 6\ldots(2n)}{1\cdot 3\cdot 5\ldots(2n-1)}\cdot\frac{1}{(n+1)}$

$$a_{n+1} = \frac{2\cdot 4\cdot 6\ldots(2n)(2n+2)}{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}\times\frac{1}{(n+2)}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{2n+1}{2n+2}\cdot\frac{n+2}{(n+1)} = 1$$

This shows D's Alembert's ratio test fail,

Now applying Raabe's test

$$\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n^2 + 4n + 2}$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \frac{1}{2} < 1$$

By Raabe's test Σa_n is divergent.

d. $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots\infty$

Here $\quad a_n = \sum\frac{n^n x^n}{n!}$

$$a_{n+1} = \frac{(n+1)^{n+1}\, x^{n+1}}{(n+1)!}$$

$$\frac{a_n}{a_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}} = \frac{n^n}{(n+1)^n} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left[\frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = \frac{1}{ex}$$

Series is convergent for $\frac{1}{ex} > 1$ *i.e.*, $ex < 1$

i.e., $x < \frac{1}{e}$

divergent for $\frac{1}{ex} < 1$ *i.e.*, $x > \frac{1}{e}$

test fails for $x = \frac{1}{e}$

Since $\frac{a_n}{a_{n+1}}$ contains e, apply Logarithmic test,

Put $x = \frac{1}{e}$ in given series.

$$\frac{a_n}{a_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$\log \frac{a_n}{a_{n+1}} = \log e \;. -n \log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \ldots\right)$$

$$\log \frac{a_n}{a_{n+1}} = \left(\frac{1}{2n} - \frac{1}{3n^2} + \ldots\right)$$

$$\lim_{n\to\infty} n \log \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} n\left[\frac{1}{2n} - \frac{1}{3n^2} + \ldots\right]$$

$$= \frac{1}{2} < 1$$

$\therefore \quad \sum a_n$ is divergent for $x = \frac{1}{e}$

Hence given series is convergent $x < \frac{1}{e}$ and divergent for $x \geq \frac{1}{e}$.

Example 3.50. *Check the convergence of* $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p + \ldots$

Solution. Neglecting 1st term

$$a_n = \left[\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\right]^p$$

$$a_{n+1} = \left[\frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots(2n)(2n+2)}\right]^p$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left(\frac{2n+2}{2n+1}\right)^p = 1$$

$\therefore$ Ratio test fails

Now apply Gauss test

$$\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{n}\right)^p\left(1+\frac{1}{2n}\right)^{-p} = \left[1+\frac{p}{n}+0\left(\frac{1}{n^2}\right)\right]\left[1-\frac{p}{2n}+0\left(\frac{1}{n^2}\right)\right]$$

$$= \left[1+\left(p-\frac{p}{2}\right)\frac{1}{n}+0\left(\frac{1}{n^2}\right)\right]$$

$$= 1 + \frac{p}{2}\cdot\frac{1}{n}+0\left(\frac{1}{n^2}\right)$$

Here $\lambda = \frac{p}{2}$

By Gauss test Σa_n is convergent if $\frac{p}{2} > 1$ *i.e.*, $p > 2$ and divergent for $p \leq 2$.

Example 3.51. *Check the convergence of the series* $\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots\infty$

Solution. Hint: $a_n = \frac{a(1+a)\ldots(n+a)}{b(1+b)\ldots(n+b)}$

Answer: Series is convergent when $b > 1 + a$ and divergent for $b \leq 1 + a$

Example 3.52. *Test the convergence of* $\sum \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$

Solution. Here

$$a_n = \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$$

$$a_{n+1} = \frac{(n+1)!}{x(x+1)(x+2)\ldots(x+n)}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{x+n}{(n+1)}\lim_{n\to\infty}\frac{n\left(1+\frac{x}{n}\right)}{n\left(1+\frac{1}{n}\right)} = 1$$

$\therefore$ D' Alembert ratio test fails.

Applying Gauss test

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \left(1+\frac{x}{n}\right)\left(1+\frac{1}{n}\right)^{-1}\\ &= \left(1+\frac{x}{n}\right)\left[1-\frac{1}{n}+0\left(\frac{1}{n^2}\right)\right]\\ &= \left[1+\frac{x-1}{n}+0\left(\frac{1}{n^2}\right)\right]\end{aligned}$$

Here $\lambda = x - 1$

By Gauss test series is convergent when $x - 1 > 1$ *i.e.*, $x > 2$ and divergent when $x - 1 \le 1$ *i.e.*, $x \le 2$.

Example 3.53. *Check the convergence of the series* $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\ldots\ldots\infty$ *(x > 0).*

Solution. Here

$$a_n = \frac{x^n}{(2n-1)(2n)}$$

$$a_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(2n+1)(2n+2)}{2n(2n-1)}\cdot\frac{1}{x}=\frac{1}{x}$$

By D' Alembert ratio test series is convergent for $x < 1$ and divergent for $x > 1$ test fails if $x = 1$

Applying Raabe's test, put $x = 1$

$$\begin{aligned}\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] &= \lim_{n\to\infty} n\left[\frac{4n^2+6n+2}{4n^2-2n}-1\right]\\ &= \lim_{n\to\infty} n\left[\frac{4n^2+6n+2-4n^2+2n}{4n^2-2n}\right]\\ &= \lim_{n\to\infty} n\left[\frac{8n+2}{4n^2-2n}\right]\\ &= \lim_{n\to\infty}\frac{8n^2+2n}{4n^2-2n}=2>1\end{aligned}$$

$\therefore$ Series is convergent.

Thus the series is convergent for $x \le 1$ and divergent for $x > 1$.

Example 3.54. *Check the convergence of the given series* $1+\frac{a}{1!}+\frac{a(a+1)}{2!}+\frac{a(a+1)(a+2)}{3!}+\ldots$

Solution. Neglecting 1st term

$$a_n = \frac{a(a+1)(a+2)\ldots(a+n-1)}{n!}$$

$$a_{n+1} = \frac{a(a+1)(a+2)\dots(a+n-1)(a+n)}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{n+1}{a+n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{a}{n}}$$

$$= 1 \qquad\qquad \therefore \quad \text{Ratio test fails.}$$

Now applying Raabe's test

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] = \lim_{n\to\infty} n\left[\frac{n+1}{a+n}-1\right]$$

$$= \lim_{n\to\infty}\frac{n(1-a)}{a+n} = \lim_{n\to\infty}\frac{1-a}{\frac{a}{n}+1}$$

$$= 1-a$$

By Raabe's test, series is convergent if $1 - a > 1$ *i.e.*, $a < 0$, divergent if $1 - a < 1$
i.e., $a > 0$

If $1 - a = 1$ *i.e.*, $a = 0$ test fails

In this case series become $1 + 0 + 0 \dots$

This is convergent.

Hence, Σa_n convergent if $a \le 0$ and divergent for $a > 0$.

Example 3.55. *Test the convergence of the series* $\sum \frac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots(2n)}\cdot x^{2n},\ x > 0.$

Solution. Try yourself.

Answer. Series is convergent if $x^2 \le 1$ and divergent for $x^2 > 1$.

Example 3.56. *Check the convergence of* $1 + \frac{(1!)^2}{2!}x^2 + \frac{(2!)^2}{4!}x^4 + \frac{(3!)^2}{6!}x^6 + \dots \infty\ (x > 0).$

Solution. Here $$a_n = \frac{(n!)^2}{(2n)!}x^{2n}$$

$$a_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!}\, x^{2n+2}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{n!\,n!}{(2n)!}\times\frac{(2n+2)(2n+1)\,2n!}{(n+1)(n+1)\,n!\,n!}\cdot\frac{1}{x^2}$$

$$= \lim_{n\to\infty}\frac{2(2n+1)}{(n+1)}\cdot\frac{1}{x^2}$$

$$= \frac{4}{x^2}$$

By Ratio test Σa_n is convergent for $\frac{4}{x^2} > 1$ *i.e.*, $x^2 < 4$ divergent for $\frac{4}{x^2} < 1$ *i.e.*, $x^2 > 4$ test fail if $x^2 = 4$.

Put $x^2 = 4$ and apply Raabe's test

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{4n+2-4n-4}{4(n+1)}\right]$$

$$= \lim_{n\to\infty} \frac{-2n}{4(n+1)} = \lim_{n\to\infty} \frac{n}{n}\left[\frac{-2}{4\left(1+\frac{1}{n}\right)}\right]$$

$$= -\frac{1}{2} < 1$$

By Raabe's test series is divergent.

Hence given series is convergent for $x^2 > 4$ and divergent for $x^2 \geq 4$.

Example 3.57. *Check the convergence of* $\frac{x^2}{2\log 2} + \frac{x^3}{3\log 3} + \frac{x^4}{4\log 4} + \ldots\infty$

Solution. Hint: $$a_n = \frac{x^{n+1}}{(n+1)\log(n+1)}$$

Answer: Series is convergent when $x < 1$ and divergent for $x \geq 1$

Example 3.58. *Check the convergence of* $1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \ldots$

Solution. Neglecting 1st term

$$a_n = \frac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)}\, x^n \; (x \geq 0)$$

$$a_{n+1} = \frac{a(a+1)(a+2)\ldots(a+n)}{b(b+1)(b+2)\ldots(b+n)} \,.\, x^{n+1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{b+n}{a+n}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

By ratio test Σa_n is convergent if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and is divergent for $\frac{1}{x} < 1$ *i.e.,* $x > 1$ and for $x = 1$ test fails.

Now apply Raabe's test for $x = 1$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = \lim_{n\to\infty} \frac{(b-a)n}{a+n}$$

$$= \lim_{n\to\infty} \frac{b-a}{\left(1+\frac{a}{n}\right)} = (b-a)$$

By Raabe's test Σa_n is convergent if $b - a > 1$ and is divergent if $b - a < 1$ *i.e.,* $b < 1 + a$

For $b - a = 1$, Apply Gauss test

$$\frac{a_n}{a_{n+1}} = \frac{b+n}{a+n} = \frac{a+1+n}{a+n}$$

$$= \frac{a+n+1}{a+n} = \frac{n\left[1+\frac{a+1}{n}\right]}{n\left[1+\frac{a}{n}\right]}$$

$$= \left[1+\left(\frac{a+1}{n}\right)\right]\left[1-\left(\frac{a}{n}\right)+\left(\frac{a}{n}\right)^2 \ldots\right]$$

$$= 1 + \left(\frac{a+1}{n}\right) - \frac{a}{n} + 0\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{1}{n} + 0\left(\frac{1}{n^2}\right)$$

By Gauss test Σa_n is divergent if $x = 1$ and $b - a = 1$.

Hence given series Σa_n is convergent when $x < 1$ and is divergent when $x > 1$.

For $x = 1$ if $b - a > 1$ it is convergent and it is divergent for $b - a \le 1$.

EXERCISE 3.8

Discuss the convergence and divergence of the following series:

1. $\frac{1^2}{4^2} + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \ldots \infty$

2. $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} x + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} x^2 + \ldots \infty$

3. $1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \ldots \infty$

4. $\frac{a}{b} + \frac{a(a+d)x}{b(b+d)} + \frac{a(a+d)(a+2d)x^2}{b(b+d)(b+2d)} + \ldots$

5. $1 + \frac{1}{2} x + \frac{1 \cdot 3}{2 \cdot 4} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \ldots$

6. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \ldots \infty$

7. $\sum \frac{4 \cdot 7 \cdot 10 \ldots (3n+1)x^n}{1 \cdot 2 \cdot 3 \ldots n}$

8. $\frac{1}{2} x + x^2 + \frac{9x^3}{8} + x^4 + \frac{25x^5}{32} + \ldots \infty$

9. $\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \ldots$

10. $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 . 4^2} + \frac{1^2 . 3^2 . 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$

11. $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots$

12. $1 + \frac{x}{2} + \frac{2!}{3^2} x^2 + \frac{3!}{4^3} x^3 + \frac{4!}{5^4} x^4 + \ldots$

13. $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \ldots$

14. $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots$

15. $1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{a(a+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \ldots$

16. $1+\frac{2}{3}x+\frac{2\cdot 3}{3\cdot 5}x^2+\frac{2\cdot 3\cdot 4}{3\cdot 5\cdot 7}x^3+\ldots$

17. $1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3+\ldots\infty$

18. $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n$

19. $\sum\frac{n^2(n+1)^2}{n!}$

20. $\frac{1}{(\log 2)^k}+\frac{1}{(\log 3)^k}+\frac{1}{(\log 4)^k}\ \ldots\infty$

Answers

1. Convergent
2. Convergent for $x < 1$, divergent for $x \geq 1$.
3. Convergent for $x \leq \frac{1}{e}$, divergent for $x > \frac{1}{e}$
4. Convergent for $x < 1$ or $x = 1$ and $b > a + d$, divergent for $x > 1$ or $x = 1$ and $b \leq a + d$.
5. Convergent for $x < 1$, divergent for $x \geq 1$.
6. Convergent for $x \leq 1$, divergent for $x > 1$.
7. Convergent for $x < \frac{1}{3}$, divergent for $x \geq \frac{1}{3}$
8. Convergent for $x < 2$, divergent for $x \geq 2$.
9. Divergent
10. Divergent
11. Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.
12. Convergent if $x < e$ and divergent if $x \geq e$.
13. Converges for $x < \frac{1}{e}$ and divergent for $x \geq \frac{1}{e}$
14. Convergent for $x < 1$ and divergent for $x \geq 1$.
15. Converges if $x < 1$ and diverges if $x > 1$.
    If $x = 1$, then the series converges if
    $\gamma > \alpha + \beta$ and diverges if $\gamma \leq \alpha + \beta$
16. Convergent for $x < 2$, divergent for $x \geq 2$.
17. Convergent for $x < 1$, divergent for $x \geq 1$.
18. Convergent for $x < \frac{1}{4}$, divergent for $x \geq \frac{1}{4}$.
19. Convergent
20. Divergent.

3.3.11 Cauchy's Integral Test

If $f(x)$ is a non-negative, monotonically decreasing and integrable function of $x \geq 1$ such that $f(n) = a_n$, $\forall\ n \in N$, then the series Σa_n and the integral $\int_1^{\infty} f(x)$ converges or diverges together.

> **Note:** If $x \geq \kappa$, then Σa_n and $\int_{\kappa}^{\infty} f(x)\,dx$ converges or diverges together.

SOME SOLVED EXAMPLES

Example 3.59. *Using integral test, discuss the convergence of the following series:*

i. $\sum \frac{1}{2n+3}$ ii. $\sum \frac{1}{n(n+1)}$ iii. $\sum \frac{1}{\sqrt{n}}$

iv. $\sum \frac{1}{(n+1)^2}$ v. $\sum \frac{2n^3}{n^4+3}$ vi. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Solution. i. Let $a_n = \frac{1}{2n+3} = f(n)$

$f(x) = \frac{1}{2x+3}$ for $x \geq 1, f(x)$ is +ve and monotonic decreasing.

∴ Cauchy integral test is applicable.

$$= \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{2x+3}\,dx$$

$$= \frac{1}{2}[\log(2x+3)]_1^{\infty} = \frac{1}{2}[\log\infty - \log 5]$$

$$= \infty \qquad \therefore \quad \text{divergent}$$

ii. $\sum \frac{1}{n((n+1))}$

Let $a_n = \frac{1}{n(n+1)} = f(n)$

$f(x) = \frac{1}{x(x+1)}$ for $x \geq 1, f(x)$ is +ve and monotonic decreasing.

∴ by Cauchy's integral test.

$$= \int_1^{\infty} \frac{1}{x(x+1)}\,dx = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= [\log x - \log(1+x)]_1^{\infty} = \left[\log\left(\frac{x}{1+x}\right)\right]_1^{\infty}$$

$$= \left[\log 1 - \log\left(1 + \frac{1}{x}\right)\right]_1^{\infty} = \left[\log\left(\frac{1}{1+\frac{1}{x}}\right)\right]_1^{\infty}$$

$$= \log 1 - \log 1 + \log 2$$

$$= \log 2 \text{ (Finite)} \quad \Rightarrow \quad \int_1^{\infty} f(x)\,dx \text{ converges and}$$

∴ given series is convergent.

iii. Try yourself.

Answer: Divergent.

iv. $\sum \frac{1}{(n+1)^2}$

Solution. Try yourself. **Answer:** convergent.

v. $\sum \frac{2n^3}{n^4+3}$

Solution. Try yourself. **Answer:** divergent.

vi. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Let
$$a_n = \frac{1}{n\sqrt{n^2-1}} = f(n)$$
$$f(x) = \frac{1}{x\sqrt{x^2-1}}$$

For $x \geq 2$, $f(x)$ is (+)ve and monotonic decreasing.

$\therefore$ Cauchy's integral test is applicable.

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} = [\sec^{-1} x]_2^{\infty}$$
$$= \left[\cos^{-1}\frac{1}{x}\right]_2^{\infty} = \left[\cos^{-1} 0 - \cos^{-1}\left(\frac{1}{2}\right)\right]$$
$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \text{ (Finite)}$$

$\int_2^{\infty} f(x)\,dx$ is convergent and hence Σa_n is also convergent.

Example 3.60. *Discuss the convergence of the following series using integral test:*

i. $\sum n \,.\, e^{-n^2}$

ii. $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^p}$, $p > 0$.

iii. $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^2}$

iv. $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$

Solution. i. Let $a_n = n \,.\, e^{-n^2} = f(n)$
$$f(x) = x \,.\, e^{-x^2}$$

For $x \geq 1$, $f(x)$ is (+) ve and monotonic decreasing.

$\therefore$ Cauchy's integral test is applicable.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x \,.\, e^{-x^2} dx$$
$$= \frac{1}{2}\int_1^{\infty} e^{-t} dt = \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_1^{\infty}$$

$$\left[\begin{aligned} &\text{Put } x^2 = t \\ &2xdx = dt \\ &xdx = \frac{dt}{2} \end{aligned}\right.$$

$$= -\frac{1}{2}\left[e^{-\infty} - \frac{1}{e}\right] = \frac{1}{2e} \quad \text{(Finite)}$$

$\therefore \quad \int_1^\infty f(x)\,dx$ is convergent, hence Σa_n is also convergent.

ii. Let $\quad f(n) = a_n = \dfrac{1}{n \log n\,(\log\log n)^p},\; p \geq 0$

$\therefore \quad f(x) = \dfrac{1}{x \log x\,(\log\log x)^p}$

For $x \geq 3$, $p > 0$, $f(x)$ is (+) ve and monotonic decreasing.

$\therefore$ Cauchy's integral test is applicable.

Case I: When $p > 1$, $\quad I_n = \displaystyle\int_3^n \frac{1}{x \log x\,(\log\log x)^p}\,.\,dx$ as $n \to \infty$

Let $\quad \log(\log x) = t$

$$\frac{1}{\log x}\cdot\frac{1}{x}\,dx = dt$$

So $\quad I_n = \displaystyle\int_3^n \frac{1}{t^p}\,dt$

$$= \left[\frac{t^{-p+1}}{-p+1}\right]_3^n = \left[\frac{(\log(\log x))^{-p+1}}{-p+1}\right]_3^n$$

$$= \frac{1}{-p+1}\left[\lim_{n\to\infty}(\log\log n)^{-p+1} - (\log\log 3)^{-p+1}\right]$$

$$= \frac{1}{1-p}\left[(\log\log\infty)^{-p+1} - (\log\log 3)^{-p+1}\right]$$

$$= \frac{1}{1-p}\left[(\infty)^{-\text{ve}} - \frac{1}{(\log\log 3)^{p-1}}\right] = \frac{-1}{1-p}\left[\frac{1}{(\log\log 3)^{p-1}}\right]$$

$$= \text{Finite}$$

$\therefore$ convergent for $p > 1$

Case II: For $p < 1$

$$I_n = \frac{1}{1-p}\,[(\infty)^{+\text{ve}} - \log(\log 3)^{1-p}] = \infty, \qquad \text{divergent for } p < 1.$$

Case III. For $p = 1$

$$I_n = \lim_{n\to\infty}\int_3^n \frac{1}{x \log x\,(\log\log x)}\,dx \qquad \left[\begin{array}{l}\text{Let } \log\log x = t \\ \dfrac{1}{\log x}\cdot\dfrac{1}{x}\,dx = dt\end{array}\right.$$

$$= \int \frac{1}{t}\,dt = \log t$$

$$= \lim_{n\to\infty}[\log\log\log 3]_3^n$$

$$= \lim_{n \to \infty} [\log \log \log n - \log \log \log 3]$$

$$= \infty \quad \text{divergent.}$$

$\therefore$ Series is convergent for $p > 1$ and divergent for $0 < p < 1$

iii. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

Solution. Try yourself.

Answer. Convergent

iv. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$

$$a_n = \frac{1}{(n+1)\log(n+1)} = f(n)$$

$$f(x) = \frac{1}{(x+1)\log(x+1)}$$

For $x \geq 1, f(x)$ is (+ve) and monotonic decreasing.

$\therefore$ Cauchy's integral test is applicable.

$$\int_1^{\infty} \frac{1}{(x+1)\log(x+1)}\,dx = [\log(\log(x+1))]_1^{\infty}$$

$$= \log\log(\infty + 1) - \log\log 2$$

$$= \infty \quad \therefore \quad \text{divergent.}$$

$\int_1^{\infty} f(x)\,dx$ is divergent and hence Σa_n is also divergent.

EXERCISE 3.9

Using integral test, test the convergence or divergence of the following series:

1. $\displaystyle\sum \frac{n}{(n^2+1)^2}$ **2.** $\displaystyle\sum \frac{1}{n^2+1}$ **3.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, (p > 0)$ **5.** $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$

Answers

1. Convergent **2.** Convergent

3. Converges if $p > 1$ and diverges if $0 < p \leq 1$. **4.** Converges if $p > 1$ and diverges if $0 < p \leq 1$.

5. Convergent

3.3.12 Alternating Series

A series of the form $a_1 - a_2 + a_3 - a_4 + \ldots$ where $a_n > 0$ for all $n \in N$ is called an alternating series, and is denoted by $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

For example:

1. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$

2. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}$

3.3.12.1 Leibnitz's Test for convergence of Alternating series

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\ a_n = a_1 - a_2 + a_3 - a_4 + \ldots\ (a_n > 0 \text{ for all } n)$ is convergent

i. $a_{n+1} < a_n$ for all n

ii. $\lim_{n\to\infty} a_n = 0$

Remarks:

a. If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is oscillating.

b. The alternating series will not be convergent if any one of the two conditions is not satisfied.

SOME SOLVED EXAMPLES

Example 3.61. *Discuss the convergence of the following series:*

a. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\infty$

b. $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\ldots\infty$

c. $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{5}+1}+\ldots\infty$

Solution. a. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\infty$

Here $\sum(-1)^{n-1}\ .\ a_n = \sum(-1)^{n-1}\cdot\frac{1}{\sqrt{n}}$

$$a_n = \frac{1}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}}$$

$$a_n - a_{n+1} = \frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} > 0$$

or $\sqrt{n} < \sqrt{n+1}$

$$\frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{n}}$$

This shows $a_{n+1} < a_n \ \forall \ n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$ thus both conditions of Leibnitz's test are satisfied.

Hence the given series is convergent.

b. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots \infty$

Here $$\sum (-1)^{n-1} a_n = \sum (-1)^{n-1} \frac{1}{\log (n+1)}$$

$$a_n = \frac{1}{\log(n+1)}$$

$$a_{n+1} = \frac{1}{\log(n+2)}$$

$$a_n - a_{n+1} = \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)}$$

$$= \frac{\log(n+2) - \log(n+1)}{\log(n+1)\log(n+2)}$$

since $\log(n+2) > \log(n+1)$

this shows $a_n - a_{n+1} > 0$

$a_{n+1} < a_n$ for all n

and $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\log (n+1)} = \frac{1}{\infty} = 0$$

Hence given series is convergent by Leibnitz test.

c. $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \ldots$

Let $\sum (-1)^{n+1} \frac{1}{\sqrt{n+1}+1} = \sum (-1)^{n-1} a_n$

$$a_n = \frac{1}{\sqrt{n+1}+1}$$

$$a_{n+1} = \frac{1}{\sqrt{n+2}+1}$$

$$n+2 > n+1 \ \forall \ n$$

$$\sqrt{n+2} > \sqrt{n+1} \text{ for all } n$$

$$\sqrt{n+2}+1 > \sqrt{n+1}+1 \text{ for all } n$$

$$\frac{1}{\sqrt{n+2}+1} < \frac{1}{\sqrt{n+1}+1} \text{ for all } n$$

$$a_{n+1} < a_n \text{ for all } n$$

Also $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+1} = 0$$

Hence the given series is convergent.

Example 3.62. *Discuss the convergence of the following series:*

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots\infty \qquad (a > 0, b > 0)$$

Solution. Let $\sum(-1)^{n-1}a_n = \sum(-1)^{n-1}\cdot\frac{1}{a+(n-1)b}$

$$a_n = \frac{1}{a+(n-1)b}$$

$$a_{n+1} = \frac{1}{a+nb}$$

$$nb > (n-1)\, b \text{ for all } n.$$

$$a + nb > a + (n-1)\, b \text{ for all } n.$$

$$\frac{1}{a+nb} < \frac{1}{a+(n-1)b} \text{ for all } n.$$

$\Rightarrow$ $$a_{n+1} < a_n \text{ for all } n.$$

Also $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{a+(n-1)b} = \frac{1}{\infty} = 0$$

$\Rightarrow$ by Leibnitz test, given series is convergent.

Example 3.63. *Check the convergence of*

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots\infty \qquad (0 < x < 1)$$

Solution. Try yourself.

Answer: Convergent.

Example 3.64. *Check the convergence of the given series:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-2}$$

Solution. Try yourself.

Answer: Convergent.

Example 3.65. *Test the convergence of the given series*

$$\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$$

Solution. Let $$a_n = \frac{n}{5^n}$$

$$a_{n+1} = \frac{(n+1)}{5^{n+1}}$$

$$a_n - a_{n+1} = \frac{n}{5^n} - \frac{(n+1)}{5^{n+1}} = \frac{5n-n-1}{5^{n+1}}$$

$$= \frac{4n-1}{5^{n+1}} > 0 \ \forall\ n \geq 1$$

$$a_n > a_{n+1}$$

$\Rightarrow$ $\qquad a_{n+1} < a_n$ for all n

and $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{5^n} = \frac{\infty}{\infty}$ from

Using L's Hospital rule.

$$= \lim_{n\to\infty} \frac{1}{5^n \log 5} = \frac{1}{\infty} = 0$$

both conditions of Leibnitz test are satisfy.

$\therefore$ given series is convergent.

Example 3.66. *Check the convergence of the given series*

$$\sum_{n=1}^{\infty} \frac{cosn\pi}{n^2+1}$$

Solution. Given series is

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum (-1)^n \frac{1}{n^2+1}$$

$$= \Sigma(-1)^n \,.\, a_n$$

$$a_n = \frac{1}{n^2+1},\ a_{n+1} = \frac{1}{(n+1)^2+1}$$

$$n + 1 > n$$

$$(n+1)^2 > n^2$$

$$(n+1)^2 + 1 > n^2 + 1$$

$$\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$$

$$a_{n+1} < a_n \text{ for all } n.$$

Also $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = \frac{1}{\infty} = 0$

$\therefore$ by Leibnitz test given series is convergent.

EXERCISE 3.10

Discuss the convergence or divergence of the following series:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

2. $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots$

3. $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} . n}{2n-1}$

5. $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} \ldots \infty$

6. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty$

7. $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \ldots \infty$

8. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} . x^n}{n(n-1)}, 0 < x < 1$

9. $\sum_{n=1}^{\infty} (-1)^{n-1} . \frac{n}{n^2+1}$

10. $\log\left(\frac{1}{2}\right) - \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) - \log\left(\frac{4}{5}\right) + \ldots \infty$

Answers

1. Convergent
2. Not convergent
3. Convergent
4. Not convergent
5. Convergent
6. Convergent
7. $\sum_{n=2}^{\infty} \frac{n(-1)^{n-1}}{5n+1}$, not convergent
8. Convergent
9. Convergent
10. Convergent

3.3.13 Absolute Convergence of a Series

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

OR

If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an absolutely convergent series.

For Example:

1. The series $\Sigma a_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ is absolutely convergent, as

   $\Sigma | a_n | = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ is a convergent series by p-test.

2. The series $\Sigma a_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$ is absolutely convergent, since

   $\Sigma | a_n | = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ being a geometric series with common ratio $\frac{1}{2} < 1$ is convergent.

3.3.14 Conditional Convergence

A series $\sum_{n=1}^{\infty} a_n$ is said to be conditionally convergent, if

i. $\sum_{n=1}^{\infty} a_n$ is convergent, but

ii. $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent.

(*i.e.*,) A series is said to be conditionally convergent if it is convergent but does not converge absolutely.

For example:

The series $\sum_{n=1}^{\infty} a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ ... is conditionally convergent series, since

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ is convergent by Leibnitz's test

where $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-test.

Hence $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series.

An important Observation: Every absolutely convergent series is convergent, but converse need not be true.

Remarks:

a. The divergence of $\Sigma|a_n|$ does not imply the divergence of Σa_n.

For example, $\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n}$ is divergent,

where as $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

b. Since $\sum_{n=1}^{\infty} |a_n|$ is a series of positive terms,

$\therefore$ all the test used for testing the convergence of positive term series are also applicable for testing the absolute convergence of the alternating series.

However these tests cannot give any information about the convergence of the alternating series.

SOME SOLVED EXAMPLES

Example 3.67. *Test the convergence and absolute convergence of the following series:*

a. $\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots$

b. $1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots (p>0)$

c. $\sum_{n=3}^{\infty} (-1)^{n-1} \sin\frac{1}{n}$

d. $\sum_{n=3}^{\infty} \frac{(-1)^{n-1} \cdot n}{5n-7}$

e. $1-\frac{1}{4\cdot 3}+\frac{1}{4^2\cdot 5}-\frac{1}{4^3\cdot 7}+\ldots$

Solution. a. $\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots\infty$

Given series is alternating series and each term is numerically less than the preceeding term

$$a_n = \Sigma(-1)^{n-1} \cdot \frac{1}{2n+3}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+3} = \frac{1}{\infty} = 0$$

By Leibnitz test given series is convergent.

$$\Sigma\,|a_n| = \frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\ldots$$

$$a_n = \frac{1}{2n+3} = f(n)$$

$$f(x) = \frac{1}{2x+3}$$

For $x \geq 1, f(x)$ is (+) ve and monotonic decreasing

$\therefore$ Cauchy integral test is applicable

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{2x+3}\,dx$$

$$= \frac{1}{2}[\log(2x+3)]_1^{\infty}$$

$$= \frac{1}{2}\,[\log\infty - \log 5] = \infty$$

By integral test $\Sigma|a_n|$ is divergent.

Hence given series is conditionally convergent.

b. $1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots\ (p>0)$

Try yourself.

Answer: Given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p < 1$.

c. $\sum_{n=1}^{\infty}(-1)^{n-1}\sin\frac{1}{n}$

Try yourself.

Answer: Given series is conditionally convergent.

d. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,.\,n}{5n-7}$

Try yourself.

Answer: Given series is absolutely convergent.

e. $1-\dfrac{1}{4\cdot 3}+\dfrac{1}{4^2\cdot 5}-\dfrac{1}{4^3\cdot 7}+\ldots$

Try yourself.

Answer: Given series is absolutely convergent.

Example 3.68. *Test the convergence of the series:*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n+2}{2^n+5}$$

Solution. Try yourself.

Answer: Given series is absolutely convergent.

Example 3.69. *Test whether the series* $1+\dfrac{1}{2^2}-\dfrac{1}{3^2}-\dfrac{1}{4^2}+\dfrac{1}{5^2}+\dfrac{1}{6^2}-\dfrac{1}{7^2}-\dfrac{1}{8^2}+\ldots$ *is convergent or not?*

Solution. Try yourself.

Answer: Series is absolutely convergent and hence convergent.

Example 3.70. *Prove that the following series converge absolutely:*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin n\alpha}{n^2},\ \alpha \text{ real}$$

Solution. Try yourself.

Answer: Given series is absolutely convergent.

Example 3.71. *Check the convergence of the series*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos^2 n\alpha}{n\sqrt{n}},\ \alpha \text{ real}$$

Solution. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos^2 n\alpha}{n\sqrt{n}}, = \sum_{n=1}^{\infty}a_n$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos^2 n\alpha}{n\sqrt{n}}$$

$\because$ $$|\cos^2 n\alpha| < 1$$

$\Rightarrow$ $$\left|\frac{(-1)^{n-1}\cos^2 n\alpha}{n\sqrt{n}}\right| < \frac{1}{n^{3/2}}$$

Now, $\dfrac{1}{n^{3/2}}$ is convergent. $\because\ p=\dfrac{3}{2}>1$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos^2 n\alpha}{n\sqrt{n}}$ is absolutely convergent.

Example 3.72. *Prove that the following series is convergent:*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3}{(n+1)!}$$

Solution. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3}{(n+1)!} = \Sigma a_n$

$$|a_n| = \frac{n^3}{(n+1)!}$$

$$|a_{n+1}| = \frac{(n+1)^3}{(n+2)!}$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n^3}{(n+1)!}\times\frac{(n+2)!}{(n+1)^3}$$

$$= \frac{n^3}{(n+1)!}\times\frac{(n+2)(n+1)!}{n^3\left(1+\frac{1}{n}\right)^3} = \frac{(n+2)}{\left(1+\frac{1}{n}\right)^3}$$

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \infty$$

$\therefore$ $|a_n|$ is convergent by D' Alembert test.

Thus given series is absolutely convergent.

Example 3.73. *For what values of x are the following series convergent:*

$$\frac{1}{1-x}+\frac{1}{2(1-x)^2}+\frac{1}{3(1-x)^3}+\ldots$$

Solution. Let $\quad a_n = \dfrac{1}{n(1-x)^n}$

and $\quad a_{n+1} = \dfrac{1}{(n+1)(1-x)^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}\left|\frac{(1-x)^n}{(1-x)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{|1-x|}$$

$\left[\because \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = l\,(\text{finite})\right.$

$l<1 \Rightarrow$ convergent

$l>1 \Rightarrow$ divergent

$\dfrac{1}{|l-x|} > 1 \Rightarrow |1-x|<1 \Rightarrow$ convergent

By ratio test $\displaystyle\sum_{n=1}^{\infty}|a_n|$ is convergent if $\dfrac{1}{|1-x|} < 1$

i.e. $\quad |(1-x) > 1$

or $\quad (1-x) > 1$ or $(1-x) < -1$

Hence for $x < 0$ or $x > 2$ series is absolutlely convergent, hence convergent.

Also for $x = 2$ the series $-1+\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{4}\ldots$ which is convergent series.

$$a_n = \frac{1}{n}$$

$$a_{n+1} = \frac{1}{n+1}$$

$$a_n > a_{n+1}$$

$\Rightarrow$ $$a_{n+1} < a_n$$

Also $$\lim_{n\to\infty} a_n = 0$$

Hence Σa_n is convergent by Leibnitz test.

EXERCISE 3.11

Test the convergence and absolute convergence of the series (1–10):

1. $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\dots$

2. $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{5}+1}+\dots$

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.n}{n^2+1}$

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{2^n}{n!}$

5. $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\dots$

6. $1-\frac{1}{2^3}(1+2)+\frac{1}{3^3}(1+2+3)-\frac{1}{4^3}(1+2+3+4)+\dots$

7. $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\log n)^2}$

8. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.n}{5^n}$

9. $\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

10. $\frac{1}{2(\log 2)^p}-\frac{1}{3(\log 3)^p}+\frac{1}{4(\log 4)^p}$ **... ∞ $(p>0)$**

11. Prove that the series

$\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}$ ... converges absolutely.

12. Discuss the convergence of the series $1+\frac{x}{1!}+\frac{x^2}{2!}+\dots$ for all value of x.

13. Discuss the convergence and absolute convergence of the series $x-\frac{x^3}{3}+\frac{x^5}{5}$ ..., x being real.

14. For what values of x are following series convergent:

a. $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\dots$

b. $1-\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^6}{6}+\dots$

15. Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^n\frac{\tan^{-1}n}{1+n^2}$.

Answers

1. Absolutely convergent

2. Conditionally convergent

3. Conditionally convergent

4. Absolutely convergent

5. Absolutely convergent
6. Conditionally convergent
7. Absolutely convergent
8. Absolutely convergent
9. Absolutely convergent
10. Absolutely convergent if $p > 1$ and Conditionally convergent if $0 < p \leq 1$.
11. Absolutely convergent for all x.
12. Converges for all values of x.
13. Given series is convergent if $-1 \leq x \leq 1$.
14. a. $-1 < x < 1$ b. $-1 < x < 1$
15. Convergent

3.4 TAYLOR'S INFINITE SERIES

If a function $f(x)$ possesses derivatives of all orders in the interval $[a, a + h]$, then for every integer h, however small, there corresponds a Taylor's expansion with Lagrange's form of remainder *i.e.*,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+ph)$$

where $0 < p < 1$

or $f(a + h) = S_n + R_n$

where $$S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

and $$R_n = \frac{h^n}{n!} f^n(a+ph)$$

considering $R_n \to 0$, as $n \to \infty$ then $S_n = f(a+h)$ as $n \to \infty$

Hence $f(a+h) = f(a) + h f'(a) + \dfrac{h^2}{2!} f''(a) + \ldots\ldots\ldots + \dfrac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \ldots$ is convergent. Thus if

i. $f(x)$ possesses derivatives of all orders in $[a, a + h]$

ii. $R_n = \dfrac{h^n}{n!} f^n(a+ph)$, tends to zero as $n \to \infty$, where, $0 < p < 1$

Then $$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a) + \ldots \quad \ldots(i)$$

Hence this is an expansion of $f(a + h)$ as an infinite series in ascending integral powers of 'h' and is called Taylor's infinite series.

Note: To express $f(x)$ in ascending integral powers of $(x - a)$, change $(a + h)$ to x *i.e.*, $h = x - a$, in equation (i), then $f(x) = f[a + (x - a)]$, we get

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) \ldots\ldots\ldots$$

3.4.1 Working Method for Expansion as Taylor's Infinite Series

To expand $f(x + h)$ as Taylor's series, we take following steps:

Step I: Let $f(x + h)$ be a given function

Step II: Put $h = 0$ and find $f(x)$

Step III: Differentiate $f(x)$ a number of times and obtain $f'(x), f''(x), f'''(x)$, ...

Step IV: Substitute the values of $f(x), f'(x), f''(x), f'''(x)$ in

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots\ldots + \frac{h^n}{n!} f^n(x) + \ldots\ldots.$$

FAILURE OF TAYLOR'S SERIES

Taylor's series fails to expand $f(x + h)$ as an infinite series in the following situations:

a. If any of functions $f(x), f'(x), f''(x)$, ... becomes infinite or does not exist, *i.e.*, indeterminate form for any value of 'x' in the interval under consideration.

b. If R_n *i.e.*, n^{th} terms does not tend to zero as $n \to \infty$.

SOME SOLVED EXAMPLES

Example 3.74. *Using Taylor's series, prove that* $e^{x+h} = e^x \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\ldots\ldots\right)$

Solution. Let $f(x + h) = e^{x+h}$

Putting $h = 0$, $f(x) = e^x$

We have, $f'(x) = e^x$, $f''(x) = e^x$, $f''' = e^x$, ...

According to Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Thus, $$e^{x+h} = e^x + e^x \,.\, h + e^x \,.\, \frac{h^2}{2!} + e^x \frac{h^3}{3!} + \ldots\ldots\ldots \quad \text{(After putting values)}$$

$$= e^x \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\ldots\right) \quad \textbf{Proved}$$

Example 3.75. *Prove that* $a^{x+h} = a^x \left(1 + h \log a + \frac{h^2}{2!} (\log a)^2 + \frac{h^3}{3!} (\log a)^3 + \ldots\ldots\ldots\right.$

Solution. Here $f(x + h) = a^{x+h}$

Putting $h = 0$, we have

$$f(x) = a^x,\ f'(x) = a^x \log a,\ f''(x) = a^x (\log a)^2,\ f'''(x) = a^x (\log a)^3$$

According to Taylor's series, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots$

Thus, $$f(x+h) = a^{x+h} = a^x + h\, a^x \log a + \frac{h^2}{2!} a^x (\log a)^2 + \frac{h^3}{3!} a^x (\log a)^3 \ldots\ldots$$

$$= a^x \left\{1 + h \log a + \frac{h^2}{2!} (\log a)^2 + \frac{h^3}{3!} (\log a)^3\right\} \ldots\ldots \textbf{ Proved.}$$

Example 3.76. *Prove that* $\log \cos(x+h) = \log \cos x - h \tan x - \frac{h^2}{2} \sec^2 x - \frac{h^3}{3} \sec^2 x \,.\, \tan x + \ldots\ldots$

Solution. Try yourself.

Example 3.77. *Prove that* $\sin^{-1}(x+h) = \sin^{-1} x + \frac{h}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}} \frac{h^2}{2!} + \ldots\ldots$

Solution. Try yourself.

Example 3.78. *Prove that* $\tan^{-1}(x+h) = \tan^{-1} x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} + \ldots\ldots\ldots$

Solution. Here $f(x+h) = \tan^{-1}(x+h)$

Putting $h = 0$, we have

$$f(x) = \tan^{-1} x,\ f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$f''(x) = (-1)(1+x^2)^{-2} \,.\, (2x) = \frac{-2x}{(1+x^2)^2}$$

According to Taylor's series, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

$$\Rightarrow \quad \tan^{-1}(x+h) = \tan^{-1} x + \frac{h}{1+x^2} - \frac{h^2 \,.\, x}{(1+x^2)^2} + \ldots\ldots\ldots$$ **Proved.**

Example 3.79. *Prove that*

$$\tan(x+h) = \tan x + h \sec^2 x + h^2 \sec^2 x \tan x + \frac{h^3}{3}(1 + 3\tan^2 x) \,.\, \sec^2 x + \ldots\ldots$$

Solution. Here $f(x+h) = \tan(x+h)$

Putting $h = 0$, we have

$$\begin{aligned} f(x) &= \tan x,\ f'(x) = \sec^2 x \\ f''(x) &= 2 \sec x \,.\, \sec x \tan x \\ &= 2 \sec^2 x \tan x \\ f'''(x) &= 2(\sec^2 x \,.\, \sec^2 x + \tan x \,.\, 2 \sec x \,.\, \sec x \tan x) \\ &= 2(\sec^2 x \,.\, \sec^2 x + 2 \sec^2 x \,.\, \tan^2 x) \\ &= 2 \sec^2 x (\sec^2 x + 2 \tan^2 x) \\ &= 2 \sec^2 x (1 + \tan^2 x + 2 \tan^2 x) \\ &= 2 \sec^2 x (1 + 3 \tan^2 x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

According to Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

$$\Rightarrow \quad \tan(x+h) = \tan x + h \,.\, \sec^2 x + \frac{h^2}{2!} \,.\, 2 \sec^2 x \,.\, \tan x + \frac{h^3}{3!} \,.\, 2 \sec^2 x (1 + 3 \tan^2 x) + \ldots\ldots$$

$$= \tan x + h \sec^2 x + h^2 \sec^2 x \tan x + \frac{h^3}{3} \sec^2 x (1 + 3 \tan^2 x) + \ldots\ldots$$ **Proved.**

Example 3.80. *Prove that* $\dfrac{f(x+h)+f(x-h)}{2} = f(x) + \dfrac{h^2}{2!} f''(x) + \dfrac{h^4}{4!} f^{iv}(x) + ...$

Solution. Try yourself.

Example 3.81. *Expand sin (x + h) in powers of h and deduce that sin (x + h) = sinx cos h + cos x sin h.*

Solution. Try yourself.

Example 3.82. *Using Taylor's series, expand e^x in powers of (x – 2).*

Solution. $f(x) = e^x = e^{2+(x-2)} = e^{2+h}$, where $h = x - 2$

$$f'(x) = e^x, \quad f(2) = e^2$$
$$f''(x) = e^x, \quad f'(2) = e^2$$
$$f'''(x) = e^x, \quad f''(2) = e^2$$
$$f'''(2) = e^2$$

According to Taylor's series,

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + ...$$

$$= e^2 + (x-2)\,.\,e^2 + \frac{(x-2)^2}{2!}\,.\,e^2 + \frac{(x-2)^3}{3!}\,e^2 + ... \qquad \text{[After putting values]}$$

$$= e^2\left[1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + ...\right]$$

Example 3.83. *Expand sin x in powers of (x – π/2). Hence find the value of sin 91° correct to four decimal places.*

Solution. Let $f(x) = \sin x$

or $= \sin[\pi/2 + (x - \pi/2)]$

$= \sin[\pi/2 + h], h = x - \pi/2$

According to Taylor's series,

$$= f(\pi/2) + hf'(\pi/2) + \frac{h^2}{2!} f''(\pi/2) + \frac{h^3}{3!} f'''(\pi/2) + ...$$

$$f(x) = \sin x, \quad f(\pi/2) = \sin \pi/2 = 1$$
$$f'(x) = \cos x, \quad f'(\pi/2) = 0$$
$$f''(x) = -\sin x, \quad f''(\pi/2) = -1$$
$$f'''(x) = -\cos x, \quad f'''(\pi/2) = 0$$
$$f^{iv}(x) = \sin x, \quad f^{iv}(\pi/2) = 1$$

$$f(x) = 1 + h\,.\,0 + \frac{h^2}{2!}(-1) + \frac{h^3}{3!}(0) + \frac{h^4}{4!}(1) + ... \qquad \text{[After putting values]}$$

$$\sin x = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} \;......$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + ... \quad \textbf{Proved Part I.}$$

Part II. Find the value of sin 91°

Solution. Let $f(x) = \sin x$

According to Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots$$

Here, let $x = 90°, h = 1°$

$$\sin(x+h) = \sin x + 1° \,.\, \cos x + \frac{(1°)^2}{2!}(-\sin x) + \frac{(1°)^3}{3!} \,.\, (-\cos x) + \ldots\ldots$$

$$\sin 91° = \sin 90° + \left(\frac{\pi}{180}\right) . \cos 90° + \left(\frac{\pi}{180}\right)^2 . \frac{1}{2}(-\sin 90°)$$

$$+ \left(\frac{\pi}{180}\right)^3 . \frac{1}{3!}(-\cos 90°) + \ldots\ldots\ldots\ldots\ldots$$

$$= 1 - \frac{1}{2}\left(\frac{\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{180}\right)^4 - \ldots\ldots\ldots\ldots$$

$$= 1 - 0.00015 + 0000000039$$

$$= 1 - 0.00015 = 0.9985$$ **Answer.**

Example 3.84. *Find $\log_e x$ in powers of $(x - 1)$. Hence evaluate $\log_e 1.1.$ correct to 4 decimal places.*

Solution. Try by yourself.

Answer. $\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots\ldots$ **Proved I Part.**

Answer: Part II. 0.99985

Example 3.85. *Expand $\tan x$ in powers of $(x - \pi/4)$.*

Solution. Try yourself.

Answer. $1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots\ldots\ldots$ **Answer**

Example 3.86. *Expand $\tan^{-1}x$ in powers of $(x - 1)$.*

Solution. Let $f(x) = \tan^{-1}[1 + (x - 1)]$

$= \tan^{-1}[x + h], \quad h = x - 1$

$= f(x + h) \quad x = 1$

$f(x) = \tan^{-1} x, \quad$ putting $h = 0$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$f''(x) = (-1)(1+x^2)^{-2}(2x)$$

$$= \frac{-2x}{(1+x^2)^2} = -2x(1+x^2)^{-2}$$

$$f'''(x) = -2\,[x \,.\, (-2)(1+x^2)^{-3}(2x) + (1+x^2)^{-2}(1)]$$

$$= 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$$

Now, we have $f(1) = \pi/4$,

$$f'(1) = \frac{1}{2}$$

$$f''(1) = \frac{-2}{4} = -\frac{1}{2}$$

$$f'''(1) = \frac{8}{8} - \frac{2}{4} = 1 - \frac{1}{2} = \frac{1}{2}$$

Putting values in Taylor's series, we have

$$f(x) = f(1) + h f'(1) + \frac{h^2}{2!} f''(1) + \ldots\ldots\ldots\ldots$$

$$= \frac{\pi}{4} + (x-1)\,.\,\frac{1}{2} + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \ldots$$

$$= \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} \ldots$$

$$= \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} \ldots \quad \textbf{Answer.}$$

Example 3.87. *Expand $2 + x^2 - 3x^5 + 7x^6$ in powers of $(x - 1)$.*

Solution. We have $f(x) = f(1 + (x - 1)) = f(x + h)$

$h = x - 1$, Put $h = 0$, we have $x = 1$

$$\begin{aligned}
f(x) &= 2 + x^2 - 3x^5 + 7x^6, & f(1) &= 2 + 1 - 3 + 7 = 7\\
f'(x) &= 2x - 15x^4 + 42x^5, & f'(1) &= 2 - 15 + 42 = 29\\
f''(x) &= 2 - 60x^3 + 210\,x^4, & f''(1) &= 2 - 60 + 210 = 152\\
f'''(x) &= -180x^2 + 840x^3, & f'''(1) &= -180 + 840 = 660\\
f^{iv}(x) &= -360x + 2520x^2, & f^{iv}(1) &= -360 + 2520 = 2160\\
f^{v}(x) &= -360 + 5040x, & f^{v}(1) &= -360 + 5040 = 4680\\
f^{vi}(x) &= 5040
\end{aligned}$$

According to Taylor's series,

$$f(x) = f(1) + h f'(1) + \frac{h^2}{2!} f''(1) + \frac{h^3}{3!} f'''(1) + \ldots$$

$$= 7 + (x - 1)\,.\,29 + \frac{(x-1)^2}{2!}\,.\,152 + \frac{(x-1)^3}{3!}\,.\,660 + \ldots \textbf{ Answer.}$$

Example 3.88. *Expand log cos x about the point π/3.*

Solution. We have $f(x) = \log\cos\left(\frac{\pi}{3} + \left(x - \frac{\pi}{3}\right)\right)$

$= \log\cos(x + h) = f(x + h)$, here $h = x - \pi/3$, $x = \pi/3$

$$\begin{aligned}
f(x) &= \log\cos x\\
f'(x) &= \frac{1}{\cos x}(-\sin x) = -\tan x\\
f''(x) &= -\sec^2 x\\
f'''(x) &= -2\sec x\,.\,\sec x\,.\,\tan x\\
&= -2\sec^2 x\,.\,\tan x
\end{aligned}$$

$$\begin{aligned}
f(\pi/3) &= \log\cos\pi/3\\
f'(\pi/3) &= \tan\pi/3 = -\sqrt{3}\\
f''(\pi/3) &= -\sec^2\pi/3 = -4\\
f'''(\pi/3) &= -2\sec^2\pi/3\,.\,\tan\pi/3\\
&= -2\,.\,(-4)\,(-\sqrt{3}) = -8\sqrt{3}
\end{aligned}$$

$$f(x) = f(\pi/3) + hf'(\pi/3) + \frac{h^2}{2!} f''(\pi/3) + \ldots\ldots$$

$$= \log\cos\frac{\pi}{3} + \left(x-\frac{\pi}{3}\right)(-\sqrt{3}) + \frac{\left(x-\frac{\pi}{3}\right)^2}{2!}(-4) + \frac{\left(x-\frac{\pi}{3}\right)^3}{3!}(-8\sqrt{3}) + \ldots$$

$$= \log(0.5) - \sqrt{3}\left(x-\frac{\pi}{3}\right) - 2\left(x-\frac{\pi}{3}\right)^2 - \frac{4\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3 + \ldots \quad \textbf{Answer.}$$

Example 3.89. *Using Taylor's series, compute the value of sin 31° to four decimal places.*

Solution. Try by yourself.

Answer. 0.5151.

Example 3.90. *If $f(x) = x^3 + 8x^2 + 15x - 24$, calculate the value of $f\left(\frac{11}{10}\right)$ by using Taylor's series.*

Solution. Try yourself.

Answer. 3.511.

Example 3.91. *Calculate the approximate value of $\sqrt{17}$ to four decimal places by taking the first four terms of an appropriate Taylor's series.*

Solution. Try yourself.

Answer. 4.123.

Example 3.92. *Prove that $f(ax) = f(x) + (a-1)x f'(x) + \frac{(a-1)^2 \cdot x^2}{2!} f''(x) + \ldots\ldots$*

Solution. According to Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots$$

We have, $\quad f(ax) = f(x + (a-1)x)$

$$= f(x+h),\ h = (a-1)x,\ x = x$$

Thus, $\quad f(ax) = f(x) + (a-1)xf'(x) + \frac{(a-1)^2 \cdot x^2}{2!} f''(x) + \frac{(a-1)^3 \cdot x^3}{3!} f'''(x) + \ldots\ldots$ **Proved.**

ii. Prove that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{f''(x)}{2!} - \left(\frac{x}{1+x}\right)^3 \frac{f'''(x)}{3!} + \ldots\ldots$$

$$f\left(\frac{x^2}{1+x}\right) = f\left(x - \frac{x}{1+x}\right)$$

$$= f(x+h)$$

$$x = x,\ h = \frac{-x}{1+x}, \text{ put values in Taylor's series,}$$

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \left(\frac{x}{1+x}\right)^2 \cdot \frac{f''(x)}{2!} - \left(\frac{x}{1+x}\right)^3 \cdot \frac{f'''(x)}{3!} + \ldots\ldots \quad \textbf{Proved.}$$

EXERCISE 3.12

1. Using Taylor's series, prove that

   a. $\dfrac{1}{x+h} = \dfrac{1}{x} - \dfrac{h}{x^2} + \dfrac{h^2}{x^3} - \dfrac{h^3}{x^4} + \ldots\ldots$

   b. $\sec^{-1}(x+h) = \sec^{-1}x + \dfrac{h}{x\sqrt{x^2-1}} - \dfrac{2x^2-1}{x^2(x^2-1)^{3/2}}\dfrac{h^2}{2!} + \ldots\ldots$

2. Using Taylor's series,

   a. Find the value of $f\left(\dfrac{21}{20}\right)$, if $f(x) = x^3 - 6x^2 + 7$

   b. Calculate the approximate value of $\sqrt{10}$ and $\sqrt{26}$ to four decimal places by taking first three terms of an appropriate Taylor's expansion.

   c. Calculate the value of $f\left(\dfrac{9}{10}\right)$ by the application of Taylor's series for $f(x+h)$, if

   $$f(x) = x^3 + 2x^2 - 5x + 11$$

3. Expand log sin x in ascending power of $(x - 3)$.
4. Calculate approximately $\log_{10} 404$, if $\log_{10} 4 = 0.6021$, using Taylor's series.
5. By use of Taylor's series, compute the value of cos 32°.
6. Use Taylor's Theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in power of $(x - 2)$.
7. Find the expansion of log sin $(x + h)$ by applying Taylor's series.

Answers

2. a. 2.1623, 5.099 b. 8.849

3. $\log\sin(3) + (x-3)\cot(3) - \dfrac{(x-3)^2}{2}\operatorname{cosec}^2(3) + \dfrac{(x-3)^3}{3}\operatorname{cosec}^3(3)\cot(3) + \ldots\ldots\ldots$

4. $\log_{10} 404 = 2.6064$ **5.** 0.8461 **6.** $40 + 53(x-2) + 19(x-2)^2 + 2(x-3)^3$

7. $\log\sin x + h\cot x - \dfrac{h^2}{2}\operatorname{cosec}^2 x + \dfrac{h^3}{3}\operatorname{cosec}^2 x\cot x + \ldots\ldots\ldots$

3.5 MACLAURIN'S INFINITE SERIES

If we put $a = 0$ and $h = x$ in Taylor's infinite series, such that

i. $f(x)$ possesses derivatives of all order in the interval $[0, x]$ and

ii. Maclaurin's remainder $R_n = \dfrac{x^n}{n!} f^n(a + ph)$ tends to zero as $n \to \infty$, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots,$$

which is known as Maclaurin's infinite series.

Note: If the function $f(x)$ denoted by y, then Maclaurin's infinite series may be written as

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots\ldots + \frac{x^n}{n!}y_n(0) + \ldots\ldots$$

where $y(0), y_1(0), y_2(0), y_3(0), \ldots\ldots y_n(0)$ etc. denote the values of $y, y_1, y_2, y_3, \ldots y_n$ respectively for $x = 0$.

3.5.1 Working Method for Expansion as Maclaurin's Series

Step I: Put $f(x)$ = given function

Step II: Differentiate $f(x)$, a number of times, *i.e.*, $f(x), f'(x), f''(x), f'''(x)$ .... and so on.

Step III: Put $x = 0$ in the result of step II and find $f(0), f'(0), f''(0), f'''(0)$ .... and so on.

Step IV: On substituting the values of $f(0), f'(0), f''(0), f'''(0)$, ... in Maclaurin's series

we get,
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots$$

Failure of Maclaurin's Series

Maclaurin's series fails to expand $f(x)$ in an infinite series in the following situation

a. If any of $f(x), f'(x), f''(x)$ .... becomes infinite or does not exist in the closed interval $[0, x]$

b. If R_n does not tend to zero as $n \to \infty$..

Some Useful Expansions

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots \qquad (x < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots$$

$$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots\ldots\ldots \qquad (a > 0)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$

$$\tan x = x + \frac{x^3}{3!} + \frac{2x^5}{15} + \ldots\ldots$$

$$\sin h\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots$$

$$\cos h\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^4}{40} + \frac{5x^2}{12} + \ldots\ldots$$

$$\cos^{-1} x = 1 + x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\ldots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

SOME SOLVED EXAMPLES

Example 3.93. *Expand the following functions, using Maclaurin's series:*

i. $f(x) = e^x$, $\quad f(0) = e^0 = 1$

$f'(x) = e^x$, $\quad f'(0) = 1$

$f''(x) = e^x$, $\quad f''(0) = e^0 = 1$

$f'''(x) = e^x$, $\quad f'''(0) = 1$

$f^{iv}(x) = e^x$, $\quad f^{iv}(0) = 1$

Solution. Using Maclaurin's series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots \text{ **Answer**}$$

ii. $f(x) = a^x, a > 0,$ $\quad f(0) = a^0 = 1$

Solution.

For given, $f'(x) = a^x \log a,$ $\quad f'(0) = \log a$

$f''(x) = a^x (\log a)^2$ $\quad f''(0) = (\log a)^2$

$f'''(x) = a^x . (\log a)^3$ $\quad f'''(0) = (\log a)^3$

$f^{iv}(x) = a^x . (\log a)^4$ $\quad f^{iv}(0) = (\log a)^4$

Using Maclaurin's series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$= 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots\ldots$$ **Answer**

iii. $f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x,$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{iv}(x) = \sin x$ $\quad f^{iv}(0) = 0$

$f^{v}(x) = \cos x$ $\quad f^{v}(0) = 1$

$f^{vi}(x) = -\sin x \qquad f^{vi}(0) = 0$

Solution. Using Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) + ...$$

$$= 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ ...... } \textbf{Answer}$$

iv. $f(x) = \cos x$

Solution. Try yourself

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{...... } \textbf{Answer}$$

v. $f(x) = \sec x$

Solution. Try yourself

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \text{...... } \textbf{Answer}$$

vi. $f(x) = \sin h\, x \qquad f(0) = 0$

$f'(x) = \cos h\, x \qquad f'(0) = 1$

$f''(x) = \sin h\, x \qquad f''(0) = 0$

$f'''(x) = \cos h\, x \qquad f'''(0) = 1$

$f^{iv}(x) = \sin h\, x \qquad f^{iv}(0) = 0$

$f^{v}(x) = \cos h\, x \qquad f^{v}(0) = 1$

Solution. According to Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ...$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{...... } \textbf{Answer}$$

vii. $f(x) = \cos h\, x$ Try yourself. **Answer:** $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

viii. $f(x) = \log(1 - x) \qquad f(0) = 0$

$f'(x) = \frac{1}{1-x}(-1) = \frac{-1}{1-x} \qquad f'(0) = \frac{-1}{1} = -1$

$f''(x) = \frac{-1}{(1-x)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1-x)^3}(-1) = \frac{-2}{(1-x)^3}, \quad f'''(0) = -2$

$$f^{iv}(x) = \frac{6}{(1-x)^4}(-1) = \frac{-6}{(1-x)^4} \qquad f^{iv}(0) = -6$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Solution. According to Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$= 0 + x(-1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(-6) + \ldots\ldots$$

$$= -x - \frac{x^2}{2!} - \frac{x^3}{3} - \frac{x^4}{4} \ \ldots\ldots$$

Example 3.94. *Prove that*

i. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots..$

Solution.

$$f(x) = e^{\sin x}$$

$$f'(x) = e^{\sin x} . \cos x$$

$$f''(x) = e^{\sin x}(-\sin x) + \cos x . e^{\sin x} . \cos x$$

$$= e^{\sin x}(\cos^2 x - \sin x)$$

$$f'''(x) = e^{\sin x}[2\cos x(-\sin x) - \cos x] + (\cos^2 x - \sin x) . e^{\sin x} . \cos x$$

$$= e^{\sin x}[-2\sin x \cos x - \cos x + \cos^3 x - \sin x \cos x]$$

$$\ldots\ldots$$

$$f(0) = 1, f'(0) = 1$$

$$f''(0) = 1, f'''(0) = 0, f^{iv}(0) = -3$$

According to Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^4}{4!}(-3) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\ldots$$ **Proved.**

ii. *Prove that* $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots\ldots$

Solution. Try yourself.

iii. *Prove that* $\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots.$

Solution. Try yourself.

iv. *Prove that* $\frac{e^x}{\cos x} = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!}$

Solution. Try yourself.

v. *Prove that* $\log \sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \ldots\ldots$

Solution.

$$f(x) = \log \sec x$$

$$f'(x) = \frac{1}{\sec x} \sec x \,.\, \tan x = \tan x$$

$$f''(x) = \sec^2 x$$

$$f'''(x) = 2 \sec^2 x \,.\, \tan x$$

$$f^{iv}(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$

$$f^{v}(x) = 4 \sec^2 x \tan x + 12 \tan x \sec^4 x + 6 \tan^3 x \ (2 \sec^2 x)$$

$$= 4 \sec^2 x \tan x + 12 \tan x \sec^4 x + 12 \tan^3 x \sec^2 x$$

$$f^{vi}(x) = 8 \sec^2 x \tan^2 x + 4 \sec^4 x + 12 \sec^6 x + 48 \tan^2 x \,.\, \sec^4 x + 36 \tan^2 x \sec^4 x + 24 \tan^4 x \sec^2 x$$

Now,

$$f(0) = \log \sec 0 = 0$$

$$f'(0) = \tan 0 = 0$$

$$f''(0) = \sec^2 0 = 1$$

$$f'''(0) = 0$$

$$f^{iv}(0) = 2$$

$$f^{v}(0) = 0$$

$$f^{vi}(0) = 16$$

Using Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$\log \sec x = \frac{x^2}{2!} + (2)\frac{x^4}{4!} + \frac{x^6}{6!}(16) + \ldots\ldots.$$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots$$ **Proved**

vi. *Prove that* $\cos^2 x = 1 - x^2 + \dfrac{x^4}{3} \ldots$

Solution.

$$f(x) = \cos^2 x \qquad f(0) = \cos^2 0 = 1$$

$$f'(x) = -2 \cos x \sin x = -\sin 2x \qquad f'(0) = 2 \sin 0 = 0$$

$$f''(x) = -2 \cos 2x \qquad f''(0) = -2 \cos 0 = -2$$

$$f'''(x) = 4 \sin 2x \qquad f'''(0) = 4 \sin 0 = 0$$

$$f^{iv}(x) = 8 \cos 2x \qquad f^{iv}(0) = 8 \cos 0 = 8$$

$$f^{v}(x) = -16 \sin 2x \qquad f^{v}(0) = -16 \sin 0 = 0$$

Using Maclaurin's series, we get

$$\cos^2 x = 1 - \frac{2x^2}{2!} + \frac{8x^4}{4!} - \ldots\ldots$$

$$= 1 - x^2 + \frac{x^4}{3} - \ldots\ldots \quad \textbf{Proved}$$

vii. *Prove that* $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

Solution. Try yourself.

viii. *Prove that* $\tan^{-1}(1 + x) = \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} \ldots\ldots$

Solution. Let $f(x) = \tan^{-1}(1 + x)$,

$$f'(x) = \frac{1}{1+(1+x)^2}$$

$$f''(x) = \frac{-1.2\,(1+x)}{[1+(1+x)^2]^2}$$

$$f'''(x) = \frac{-2\,[\{1+(1+x)\}^2\,(1)-(1+x)\,.\,2\,(1+(1+x)^2)\,.\,2(1+x)]}{[(1+x)^2+1]^4}$$

$$\vdots$$

$$f(0) = \tan^{-1} 1 = \pi/4$$

$$f'(0) = 1/2$$

$$f''(0) = -2/4 = -1/2$$

$$f'''(0) = 1/2$$

$$\vdots$$

Using Maclaurin's series,

$$\tan^{-1}(1 + x) = \pi/4 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(-\frac{1}{2}\right) + \frac{x^3}{3!}\left(\frac{1}{2}\right) + \ldots$$

$$= \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} \ldots\ldots \textbf{ Proved}$$

Example 3.95. *Prove that*

i. $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots\ldots.$

Solution. Try yourself.

ii. *Prove that* $\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots\ldots$

Solution. Try yourself.

iii. *Prove that* $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots\ldots.$

Solution. Try yourself.

Example 3.96. *Prove that*

i. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right)$

Solution. Try yourself. **Hint:** Put $x = \tan\theta$

ii. *Prove* $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = \dfrac{1}{2}\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right)$

Solution. Try by yourself. **Hint:** Put $x = \tan\theta$

iii. *Prove that* $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}} = x + \dfrac{1^2}{3!}x^3 + \dfrac{1^2 3^2}{5!}x^5 + \ldots\ldots$

Solution. Try yourself. **Hint:** Put $x = \sin\theta$

iv. *Prove that* $\sin^{-1}(3x - 4x^3) = 3\left(x + \dfrac{x^3}{2} + \dfrac{3x^5}{40} + \ldots\ldots\right)$

Solution. Try yourself. **Hint:** Put $x = \sin\theta$

3.6 Power Series

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine the convergence and divergence of a power series. We also show how to represent certain functions using power series.

3.6.1 Definition

Power series is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \qquad \ldots(1)$$

where coefficients $c_0, c_1, c_2, \ldots$ and variable x and 'a' are all real numbers. The constant 'a' is called the centre of the power series.

In particular, when centre 'a' is zero, the power series is of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

3.6.2 Convergence of a Power Series

It is a clear that for $x = 0$, every power series is convergent, independent of the values of the coefficients. Now we give three possible cases about the convergence of a power series.

i. The series converges for no values of x, other than $x = 0$ (which is trivial point of convergence), then it is called "nowhere convergent". For example, the series $\sum_{n=1}^{\infty} n^n x^n$ converges for no value of x other than $x = 0$ and hence, is a nowhere convergent series.

ii. The series converges for all values of x, then it is called "everywhere convergent."

For example: the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!}$ are everywhere convergent.

iii. The series converges for some values of x and diverges for others.

For example: the series $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{(n+1)}}$ converges for $x \le 1$ and diverges for $x > 1$.

The collection of points x for which the series is convergent is called its "region of convergence."

3.6.3 Radius of Convergence

For every power series $\sum_{n=0}^{\infty} a_n x^n$, (except those which are everywhere convergent or nowhere convergent), there exists a finite positive number R, such that the series converges for every $|x| < R$ and diverges for every $|x| > R$.

This number R is called radius of convergence of given power series.

3.6.4 Interval of Convergence

Let R be the radius of convergence of a power series $\sum_{n=1}^{\infty} a_n x^n$. Then the open interval $(-R, R)$ is called the interval of convergence, of the given power series.

Useful Results

1. Let $\sum_{n=1}^{\infty} a_n x^n$ be a power series such that $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \frac{1}{R}$, then the power series is convergent with radius of convergence R.
2. For the power series $\sum_{n=1}^{\infty} a_n x^n$, the radius of convergence is also defined by $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$, provided the limit exists.

3.6.5 Graphical Representation of Power Series

For a series $\sum_{n=0}^{\infty} c_n (x-a)^n$ graph (a) shows a radius of convergence at $R = 0$, graph (b) shows a radius of convergence at $R = \infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x = a + R$ and $x = a - R$.

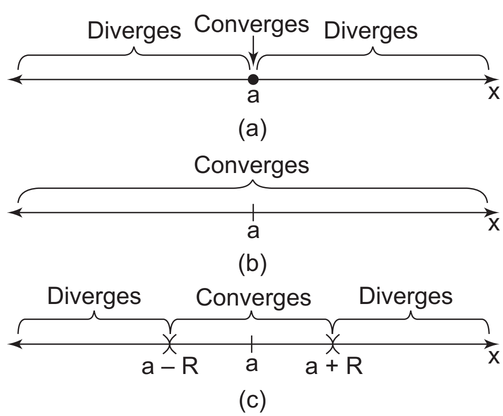


Fig. 3.4

SOME SOLVED EXAMPLES

Example 3.97. *Determine the radius of convergence of following power series:*

i. $\sum_{n=0}^{\infty} n!\,x^n$ ii. $\sum_{n=0}^{\infty} (2n-1)\,x^n$

Solution. i. Consider power series $\sum_{n=0}^{\infty} n!\,x^n$

where $a_n = n!$ with centre 0.

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)\,n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

$\therefore$ Radius of convergence *i.e.,* $R = 0$

$\Rightarrow$ Power series converges only at the centre 0.

ii. Here, $a_n = (2n-1)$ with centre 0.

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1-\frac{1}{2n}}{1+\frac{1}{2n}}\right| = 1$$

$\therefore$ Radius of convergence *i.e.,* $R = 1$

Example 3.98. *Determine radius of convergence of following series:*

i. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$ ii. $\sum_{n=0}^{\infty} (n+2)^n\,(x-4)^n$

Solution. i. Here, $a_n = \frac{1}{n!}$ with centre 2.

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right|$$

$$= \lim_{n\to\infty}|n+1| = \infty$$

$\therefore$ Radius of convergence, *i.e.,* $R = \infty$

ii. Here, $a_n = (n+2)^n$ with centre 4

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n}$$

$$= \lim_{n\to\infty} |(n+2)^n|^{1/n}$$

$$= \lim_{n\to\infty} |n+2|$$

$$= \infty$$

$\therefore$ Radius of convergence, *i.e.*, $R = 0$

Example 3.99. *Determine Radius of convergence and interval of convergence of power series* $\sum_{n=0}^{\infty} 3^{-n} x^{3n}$.

Solution. Here $a_n = 3^{-n}$

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} 3^{-1}$$

$$= \frac{1}{3}$$

$$\therefore \quad R = 3^{1/3}$$

and interval of convergence is $(-3^{1/3}, 3^{1/3})$

At $x = -3^{1/3}$, $\sum_{n=0}^{\infty} 3^{-n} [(-3)^{1/3}]^{3n}$

$$= \sum_{n=0}^{\infty} 3^{-n} (-3)^n$$

$$= \sum_{n=0}^{\infty} 3^{-n} (-1)^n (3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n, \begin{cases} 0, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow$ series is not convergent at

$$x = -3^{1/3}$$

At $x = 3^{1/3}$, $\sum_{n=0}^{\infty} 3^{-n} [(3)^{1/3}]^{3n}$

$$= \sum_{n=0}^{\infty} 3^{-n} (3)^n = \sum_{n=0}^{\infty} (3)^{-n+n}$$

$= 1$, which is divergent

so the interval of convergence is $(-3^{1/3}, 3^{1/3})$.

Example 3.100. *Determine radius and interval of convergence of power series* $\sum_{n=0}^{\infty} \frac{nx^n}{(n+1)^2}$

Solution. Here $a_n = \frac{n}{(n+1)^2}$ with centre 0

$$\therefore \quad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n(n+2)^2}{(n+1)(n+1)^2}\right|$$

$$= \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)\left(\frac{n+2}{n+1}\right)^2\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{1+2/n}{1+1/n}\right)^2\right|$$

$$= 1$$

$\therefore$ Radius of convergence = 1 and series is convergent inside the interval $|x - 0| < 1$ *i.e.*, $-1 < x < 1$

At $x = -1$ power series become

$\sum_{n=0}^{\infty}\frac{n}{(n+1)^2}(-1)^n$ which is an alternating series and convergent by Leibnitz's test as

$$a_n > a_{n+1} \text{ and } \lim_{n\to\infty}\frac{n}{(n+1)^2} = 0$$

At $x = 1$ power series becomes $\sum_{n=0}^{\infty}\frac{n}{(n+1)^2}$

which is positive series and is divergent as

$$a_n = \frac{n}{(n+1)^2} = \frac{1}{n\left(1+\frac{1}{n}\right)^2}$$

Let $\qquad b_n = \frac{1}{n}$, so that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^2} = 1 \text{ [non-zero, finite]}$$

$\therefore$ Σa_n and Σb_n converge and diverge together

But $\qquad \Sigma b_n = \sum\frac{1}{n}$ is divergent by p-test ($p = 1$)

$\therefore \qquad \Sigma a_n = \sum\frac{n}{(n+1)^2}$ is divergent.

Hence interval of convergence of power series is $[-1, 1)$

Example 3.101. *Determine radius of convergence of power series* $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$. *Also find interval of convergence of power series.*

Solution. Here $\qquad a_n = \frac{(-1)^{n+1}}{n}$ with centre of power series is 1.

$$\therefore \qquad R = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}.(n+1)}{n.(-1)^{n+2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n.(-1)}\right| = 1.$$

$\therefore$ Radius of convergence = 1 and series is convergent inside the interval $|x - 1| < 1$ *i.e.*, $-1 < x - 1 < 1 \Rightarrow 0 < x < 2$.

At $x = 2$ power series become, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ which is an alternating series and convergent by Leibnitz's test as $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

Clearly, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$

$\therefore$ Power series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ $(x-1)^n = 0$ is convergent at $x = 2$.

At $x = 0$, power series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}(-1)}{n}$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n} \text{ which is divergent by } p\text{-test } [p = 1]$$

$\therefore$ Interval of convergence of power series = (0, 2].

EXERCISE 3.13

1. Determine the radius of convergence and the interval of converge of each of the following power series:

i. $\sum \frac{(n+1)x^n}{(n+2)(n+3)}$

ii. $\sum \frac{2^n x^n}{n!}$

iii. $\sum \frac{(n!)^2 x^{2n}}{(2n)!}$

iv. $\sum \frac{(-1)^n x^{2n}}{(n!)^2 2^{2n}}$

v. $\sum \frac{(x-1)^n}{2n}$

vi. $\sum \frac{n!(x+2)^n}{n^n}$

vii. $\sum_{n=2}^{\infty} \frac{(x+2)^n}{\log n}$

viii. $\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

2. Prove that the power series $1 + \frac{a.b}{1.c}x + \frac{a(a+1)b(b+1)}{1.2\,c\,(c+1)}x^2 + \ldots$ has unit radius of convergence.

3. Determine the radius of convergence of following power series

i. $\sum_{n=0}^{\infty} \frac{1}{2^n}(x-1)^{2n}$

ii. $\sum_{n=0}^{\infty} \frac{(x-2)^{2n}}{n!}$

iii. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n . x^{2n}$

iv. $\sum_{n=0}^{\infty} (n+1)!(x-a)^n$

v. $\sum_{n=0}^{\infty} n^n (x-5)^{2n}$

Answers

1. i. $R = 1, [-1, 1)$ ii. $R = \infty, \mathbb{R}$ iii. $R = 2, (-2, 2)$
iv. $R = \infty, \mathbb{R}$ v. $R = 2, (-1, 3)$ vi. $R = e, (-2 - e, -2 + e)$
vii. $R = 1, [-3, -1)$ viii. $R = \sqrt{2},\ (-R, R)$

3. i. $\sqrt{2}$ ii. ∞ iii. $\sqrt{\frac{3}{2}}$
iv. 0 v. 0

3.7 FOURIER SERIES

Many engineering and physical problems involve the study of fourier series for their solution; for example, conduction of heat, electrodynamics and mechanical vibrations. It is necessary to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form,

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of the values of the variable. Such a series is known as the Fourier Series. Some important definite integrals, are as follows:

a. $\int_c^{c+2\pi} \sin mx\, dx = 0$

b. $\int_c^{c+2\pi} \cos mx\, dx = 0$

c. $\int_c^{c+2\pi} \sin mx \cos nx\, dx = 0, m \neq n$

d. $\int_c^{c+2\pi} \cos mx \sin nx\, dx = 0, m \neq n$

e. $\int_c^{c+2\pi} \sin^2 mx\, dx = \pi, m \neq 0$

f. $\int_c^{c+2\pi} \cos^2 mx\, dx = \pi, m \neq 0$

g. $\int_c^{c+2\pi} \cos mx \cos nx\, dx = 0, m \neq n$

h. $\int_c^{c+2\pi} \sin mx \sin nx\, dx = 0, m \neq n$

3.7.1 Dirichlet's Conditions for Fourier Series

If a function $f(x)$ is defined in the interval $c \leq x \leq c + 2\pi$ is

i. finite, single valued in the interval.
ii. has a finite number of singularities in the interval.
iii. is periodic.
iv. has finite number of maxima and minima in the interval.

then the fourier series for the function $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

a_0, a_n, b_n are known as Euler's formulae.

The above four conditions are known as Dirichlet's conditions.

Note 1: It must be noted that the fourier expansion is not valid in every case.

Note 2: The Euler's formulae a_0, a_n, b_n are obtained as follows, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

Integrate both sides of equation (1) between the limits c to $c + 2\pi$, we have

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} \frac{a_0}{2}\,dx + \int_c^{c+2\pi} \sum_{n=1}^{\infty} a_n \cos nx\,dx + \int_c^{c+2\pi} \sum_{n=1}^{\infty} b_n \sin nx\,dx$$

$$\Rightarrow \quad \int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2} 2\pi + a_n \times 0 + b_n \times 0$$
$$= a_0\,\pi$$

or
$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \qquad \left\{\begin{array}{l}\text{Since } \int_c^{c+2\pi} \cos mx\,dx = 0,\\ \text{and } \int_c^{c+2\pi} \sin mx\,dx = 0\end{array}\right.$$

Also multiplying (1) by $\cos nx$ and integrate between the limits $(c, c + 2\pi)$, we have

$$\int_c^{c+2\pi} f(x)\cos nx\,dx$$
$$= \int_c^{c+2\pi} \frac{a_0}{2}\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

then,
$$= \frac{a_0}{2}\times 0 + a_n\,\pi + 0$$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \quad \text{as} \quad \left\{\begin{array}{l}\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx\\ = \int_c^{c+2\pi}(a_1 \cos x + a_2 \cos 2x \ldots + a_n \cos nx)\cos nx\,dx\\ = a_n\,\pi \left\{\begin{array}{r}As \int_c^{c+2\pi} \cos nx \cos mx\,dx = 0, m \neq n\\ = \pi, m = n\end{array}\right\}\end{array}\right.$$

Similarly, $$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Further

i. If $c = 0$, then the interval becomes, $0 \le x \le 2\pi$, then from equation (1),

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we have, $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,$$

and $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

ii. If $c = -\pi$, then the interval becomes $-\pi \le x \le \pi$, then using equation (1), we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, ,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

SOME SOLVED EXAMPLES

Example 3.102. *Obtain a Fourier series to represent e^{-ax} from $-\pi$ to π. Hence derive series for $\pi/\sinh \pi$.*

Solution. Let $$f(x) = e^{-ax}$$

then, $$e^{-ax} = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-a\pi}[e^{-a\pi} - e^{a\pi}] = \frac{2}{a\pi}\sinh a\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}[-a\cos nx + n\sin nx]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{-a\pi}\{-a\cos n\pi + 0\} - e^{a\pi}\{-a\cos(-n\pi) + 0\}]$$

$$= \frac{1}{\pi(a^2+n^2)}[-a(-1)^n\{e^{-a\pi}-e^{a\pi}\}]$$

$$= \frac{2a(-1)^n}{\pi(a^2+n^2)}\sin h\, a\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi[a^2+n^2]}\{e^{-a\pi}(-0-n\cos n\pi) - e^{a\pi}(0-n\cos(-n\pi)\}]$$

$$= \frac{1}{\pi(a^2+n^2)}\{(-1)^n\, n\{e^{a\pi}-e^{-a\pi}\}\}$$

$$= \frac{2(-1)^n\, n}{\pi(a^2+n^2)}\sinh a\pi$$

Thus, $$e^{-ax} = f(x) = \frac{\sinh a\pi}{\pi a} + \sum_{n=1}^{\infty}\frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^n}{\pi(a^2+n^2)}\sinh a\pi\sin nx$$

$$= \frac{\sinh a\pi}{\pi a} - \frac{2a}{\pi(a^2+1^2)}\sin h\, a\pi\cos x + \frac{2a}{\pi(a^2+2^2)}\sinh a\pi\cos 2x$$

$$- \frac{2a}{\pi(a^2+3^2)}\sin h\, a\pi\cos 3x + \ldots\ldots$$

$$- \frac{2}{\pi(a^2+1^2)}\sin h\, a\pi\sin x + \frac{2\times 2}{\pi(a^2+2^2)}\sin h\, a\pi\sin 2x$$

$$- \frac{2\times 3}{\pi(a^2+3^2)}\sin h\, a\pi\sin 3x + \ldots$$

Put, $x = 0$, $a = 1$ in above equation, we have

$$1 = \frac{\sin h\pi}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(1^2+n^2)}$$

$$= \frac{2\sin h\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi(1^2+n^2)}\right]$$

$$\frac{\pi}{\sin h\pi} = 2\left[\frac{1}{2} - \frac{1}{1^2+1^2} + \frac{1}{1^2+2^2} + \frac{1}{1^2+3^2}\ldots\ldots\right]$$

$$= 2\left[\frac{1}{1^2+2^2} - \frac{1}{1^2+3^2} + \frac{1}{1^2+4^2}\ldots\ldots\right]$$

Example 3.103. *Prove that* $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)\frac{\cos nx}{n^2}, -\pi < x < \pi.$

Hence show that

1. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

2. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \ldots\ldots = \frac{\pi^2}{12}$

3. $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

4. $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

Solution. Let $f(x) = x^2$ and

$$x^2 = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\frac{(-\cos nx)}{n^2} + 2\frac{(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi} \qquad \text{(since } \sin nx = 0\text{, between the limits 0 and } \pi)$$

$$= \frac{4}{\pi n^2}[\pi\cos n\pi - 0\times\cos(0)]$$

$$= \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0$$

(since $x^2 \sin nx$ is an odd function)

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}\cos nx \qquad \ldots(1)$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2}\ldots\ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2}\ldots\ldots\right]$$

Put $x = \pi$ in (1),
$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos n\pi\right]$$

or $$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n(-1)^n}{n^2}\right]$$

$$= 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\Rightarrow$ $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \qquad ...(2)$$

Thus, $$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}......$$

Put $x = 0$ in (1), we have

$$0 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$\Rightarrow$ $$= \frac{\pi^3}{3}+4\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-......\right]$$

$\Rightarrow$ $$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...... \qquad \text{[Part (2) proved]}$$

Adding (2) and (3), we have

$$\frac{\pi^2}{6}+\frac{\pi^2}{12} = 2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}......\right]$$

$$\frac{\pi^2}{8} = \frac{1}{(2n-1)^2} \qquad \text{[Part (3) Proved]}$$

We have, $$x^2 = \frac{\pi^3}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2} \qquad \text{[From (1)]}$$

Multiplying the above with 'x^2' on both sides, we have

$$\int_{-\pi}^{\pi}x^4\,x = \int_{-\pi}^{\pi}\frac{\pi^2}{3}x^2\,dx+4\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{x^2\cos nx}{n^2}dx$$

$\Rightarrow$ $$\frac{2\pi^5}{5} = \frac{1}{9}(2\pi^5)+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_{-\pi}^{\pi}x^2\cos nx\,dx$$

$$= \frac{2}{9}\pi^5+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.2\left[x^2\frac{\sin nx}{n}-2x\frac{(-\cos nx)}{n^2}+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{9}\pi^5+8\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{2\pi\cos n\pi}{n^2}\right)$$

$$= \frac{2}{9}\pi^5+16\pi\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^4}$$

$$= \frac{2}{9}\pi^5 + 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}$$

or $$16\pi \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{2\pi^5}{5} - \frac{2}{9}\pi^5$$

$$= \frac{8\pi^5}{45}$$

Thus, $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$ [Part (4) Proved]

Example 3.104. *Find the Fourier series for the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$. Hence deduce that* $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \ldots\ldots$

Solution. Try yourself.

Example 3.105. *Evaluate $f(x) = x \sin x$, $0 < x < 2\pi$ as Fourier series and hence deduce that*

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots$$

Solution. Given, $f(x) = x \sin x$

$\Rightarrow$ Let $$x \sin x = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then, $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx = \frac{1}{\pi}\left[x(-\cos x) - (-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi] = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x[\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left| x\left(\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right) - \left(\frac{-\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right)\right|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right)\right]$$

[2nd Integral will become zero after putting limits]

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{2}{n^2-1}\right)\right], n \neq \pm 1$$

$$= \frac{2}{n^2-1}, n \neq \pm 1$$

When $n = 1$,
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx = \frac{1}{2\pi}\left[x - \left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$$

Now,
$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos (n-1)x - \cos (n+1)x] \, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1}\right) - \left(\frac{-\cos (n-1)x}{(n-1)^2} - \frac{-\cos (n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right) - \left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right)\right]$$

$$= 0, n \neq \pm 1$$

When $n = 1$,
$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x) \, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - \left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left| x^2 - \frac{x^2}{2} - \frac{\cos 2x}{4} - \frac{x \sin 2x}{2}\right|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left|\left(\frac{4\pi^2}{2} - \frac{1}{4}\right) - \left(-\frac{1}{4}\right)\right|$$

$$= \frac{1}{2\pi} \times 2\pi^2 = \pi$$

Thus,
$$x \sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x + \sum_{n=2}^{\infty} 0 \sin nx \qquad \text{...(1)}$$

$$\Rightarrow \qquad x \sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x \qquad ...(2)$$

Put $x = \dfrac{\pi}{2}$ in (1), we have

$$\frac{\pi}{2} = -1 + \pi + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}$$

$$= -1 + \pi + 2\left[\frac{1}{2^2-1}\cos\pi + \frac{1}{3^2-1}\cos\frac{3\pi}{2} + \frac{1}{4^2-1}\cos 2\pi\right]$$

or
$$-\pi + \frac{\pi}{2} = -1 - 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ......\right]$$

$$1 - \frac{\pi}{2} = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ......\right]$$

$$\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ......$$

Example 3.106. *If* $f(x) = \left(\dfrac{\pi - x}{4}\right)^2$ *in the range* $0 < x < 2\pi$, *show that* $f(x) = \dfrac{\pi^2}{48} + \dfrac{1}{4}\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$.

Solution. Given, $f(x) = \left(\dfrac{\pi - x}{4}\right)^2$, we have

$$\left(\frac{\pi-x}{4}\right)^2 = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{4}\right)^2 dx = \frac{\pi^2}{24}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{4}\right)^2 \cos nx\,dx = \frac{1}{16\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{16\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\frac{-\cos nx}{n^2} + 2(-1)^2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{16\pi}\left[(\pi-x)^2\frac{\sin nx}{n} + 2(\pi-x)\left(\frac{-\cos nx}{n^2}\right) - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{16\pi}\left[-2(\pi-x)\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{16\pi}\left[2\pi\frac{\cos 2n\pi}{n^2} + 2\pi\frac{1}{n^2}\right]$$

$$= \frac{\pi}{16\pi}\left[\frac{2}{n^2} + \frac{2}{n^2}\right] = \frac{1}{16}\left[\frac{4}{n^2}\right] = \frac{1}{4n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{4}\right)^2 \sin nx\,dx = 0$$

Thus $$\frac{(\pi - x)^2}{4} = \frac{\pi^2}{48} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

Example 3.107. *Prove that in the range $-\pi < x < \pi$.*

$$\cosh ax = \frac{2a^2}{\pi}\sin a\pi\left\{\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2}\cos nx\right\}$$

Solution. Try yourself.

EXERCISE 3.14

1. Find Fourier series for the function $f(x) = x\cos x$, $0 < x < 2\pi$.
2. Find the Fourier series for $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$
3. Expand $f(x) = \sin ax$ in $-\pi < x < \pi$ as Fourier series.
4. Expand $f(x) = x - x^2$ in $-\pi < x < \pi$ as Fourier series.
5. Obtain Fourier series for the function $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \dfrac{2x}{\pi}, & 0 < x < \pi \end{cases}$
6. Find Fourier series expansion for $f(x) = |\sin x|$, $-\pi < x < \pi$.
7. Find Fourier series of the function, when $f(x + 2\pi) = f(x)$, where $f(x) = \begin{cases} x + \pi, & 0 \le x \le \pi \\ -x + \pi, & \pi \le x \le 2\pi \end{cases}$
8. An alternating-current after passing through a rectifier has the given form where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.
$$i = \begin{cases} I_0 \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$$
9. Expand $f(x) = |\cos x|$ in $-\pi < x < \pi$ as Fourier series.
10. Expand $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$ as Fourier series.

Answers

1. $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2 - 1}\sin nx$

2. $f(x) = \frac{-\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$

3. $\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3\sin 3x}{3^2 - a^2} - \ldots\ldots\right]$

4. $x - x^2 = \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right]$

5. $f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$

6. $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\ldots\right]$

7. $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$

8. $i = \frac{I_0}{2} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \ldots\ldots\right]$

9. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\ldots\right]$

10. $\sqrt{1 - \cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=2}^{\infty}\frac{\cos nx}{4n^2 - 1}$

3.7.2 Even and Odd Functions

If $f(-x) = f(x)$, then the function $f(x)$ is an even function. **Example:** x^2, $\cos x$.
If $f(-x) = -f(x)$, then the function $f(x)$ is an odd function. **Example:** x, $\sin x$.

When $f(x)$ is even, then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$ (even function × odd function) = odd function

When $f(x)$ is odd, then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$ (odd function × odd function) = even function

3.7.3 Fourier Series for Discontinuous Functions

Let a function $f(x)$ be discontinuous at $x = x_0$

This implies $f(x_0 - 0) < f(x_0)$ and $f(x_0 + 0) > f(x_0)$

If both the limits $f(x_0 - 0), f(x_0 + 0)$ exists but are not equal. Then the value of the function at a point of discontinuity is taken as the mean of $f(x_0 - 0), f(x_0 + 0)$.

i.e., $$f(x_0) = \frac{f(x_0 - 0) + f(x_0 + 0)}{2}$$

SOME SOLVED EXAMPLES

Example 3.108. *Find the Fourier expansion for f(x) if*

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Also deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

We know, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \Rightarrow \quad \frac{1}{\pi}\int_{-\pi}^{0} -\pi\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}(-\pi x)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \pi^2 + \frac{\pi^2}{2} - 0\right] = \frac{-\pi^2}{2}\left(\frac{1}{\pi}\right)$$

$\Rightarrow$ $$a_0 = \frac{-\pi}{2}$$

Now, $$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left(-\pi\frac{\sin nx}{n}\right)_{-\pi}^{0} + \frac{1}{\pi}\left| x\frac{\sin nx}{n} - \left(\frac{-\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}(-\pi \times 0) + \frac{1}{\pi}\left|\frac{\cos n\pi}{n^2} + \left(\frac{-1}{n^2}\right)\right|$$

$$= \frac{\cos n\pi - 1}{\pi n^2} = \frac{(-1)^n - 1}{\pi n^2} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-2}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$a_1 = -\frac{2}{1^2 . \pi} . a_2 = 0, a_3 = -\frac{2}{3^2 . \pi}\, a_4 = 0, a_5 = -\frac{2}{5^2 . \pi} \ldots\ldots$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\frac{(-\cos nx)}{n} - \frac{(-\sin nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi)\left(\frac{\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n}\right) - \pi\left(\frac{\cos n\pi}{n}\right)\right] + \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right]$$

$$= \frac{\pi}{\pi}\left[\frac{1}{n} - \frac{2(-1)^n}{n}\right]$$

$$= \frac{1}{n} - 2\frac{(-1)^n}{n} = \frac{1}{n}[1 - 2(-1)^n]$$

$$b_1 = 3, b_2 = -\frac{1}{2}, b_3 = 1, b_4 = -\frac{1}{4}$$

Thus,

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots.\right] \quad \ldots(1)$$

Putting $x = 0$, in (1), we obtain

$$f(0) = \frac{-\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right] \quad \ldots(2)$$

since $x = 0$ is the point of discontinuity,

$$\therefore \quad f(0) = \frac{f(0-0) + f(0+0)}{2} = \frac{-\pi + 0}{2} = \frac{-\pi}{2}$$

$\therefore$ By (2) $\Rightarrow$

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

or

$$\frac{-\pi}{4} = -\frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Example 3.109. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, *then prove that* $f(x) = \frac{1}{x} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2 - 1}$.

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots\ldots = \frac{\pi - 2}{4}$.

Solution. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \Rightarrow \quad \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{1}{\pi}[1+1] = \frac{2}{\pi}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \Rightarrow \quad \int_{-\pi}^{0} 0\,.\,dx + \int_0^{\pi} \sin x \cos nx\,dx$$

$$\Rightarrow \quad = \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x]\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{n-1}\right]_0^{\pi}, n \neq 1$$

$$= \frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right) - \left(-\frac{1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When n is even $n + 1$, $n - 1$ are odd, then

$$a_n = \frac{1}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{1}{\pi}\left[\frac{-2}{n^2-1}\right], n \neq 1$$

When n is odd $n + 1$, $n - 1$ are even, then

$$a_n = \frac{1}{2\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0, \quad n \neq 1$$

When $n = 1$, $a_1 = 0$ (Students can check)

Now $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$\Rightarrow$ $$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \,.\, \sin nx \, dx + \int_0^{\pi} \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} 2 \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x] \, dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0, n \neq 1$$

When $n = 1$, $$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{1}{2}$$ (Students can find)

$\therefore$ $$f(x) = \frac{1}{\pi} + \frac{1}{\pi}\sum_{2,4,6}^{\infty} \frac{-2}{n^2-1} \cos nx + \frac{1}{2} \sin x$$

or $$f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi}\sum \frac{\cos nx}{n^2-1}, \text{ where } n \text{ is even integer} \quad (3)$$

Put $n = 2m$ in (3) we will get the required result,

i.e. $$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots = \frac{\pi - 2}{4}.$$

Example 3.110. *Find Fourier series for the given* $f(x) = \begin{cases} -x^2, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$.

Solution. When $-\pi \le x \le 0$ or $0 \le -x \le \pi$

$\Rightarrow$ $$f(-x) = (-x)^2 = x^2 = -f(x)$$

When $0 \le x \le \pi, \quad -\pi \le -x \le 0$

$$f(-x) = -(-x)^2 = -x^2 = -f(x)$$

$\Rightarrow$ $f(x)$ is an odd function of x in $[-\pi, \pi]$. Also the graph of above function is symmetrical about the origin.

$$a_0 = 0 = a_n$$

Let
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

when
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2 \sin nx\, dx + \int_0^{\pi} x^2 \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x^2(-\cos nx)}{n} - \frac{(-2x)(-\sin nx)}{n^2} + \frac{(-2)(-\cos nx)}{n^3}\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[\frac{x^2(-\cos nx)}{n} - \frac{(2x)(-\sin nx)}{n^2} + \frac{(2)(\cos nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-2}{n^3}\right) - \left\{\frac{-\pi^2(-\cos n\pi)}{n} + \frac{-2\cos n(-\pi)}{n^3}\right\}\right] + \frac{1}{\pi}\left[\left(\frac{-\pi^2 \cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \left(\frac{2}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^3} - \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n} - \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4}{n^3} - \frac{2\pi^2(-1)^n}{n} + \frac{4(-1)^n}{n^3}\right]$$

$$b_1 = \frac{1}{\pi}\left[\frac{-4}{1^3} + \frac{2\pi^2}{1} - \frac{4}{1^3}\right] = \frac{1}{\pi}\left[\frac{2\pi^2}{1} - \frac{8}{1^3}\right]$$

$$b_2 = \frac{1}{\pi}\left[\frac{-4}{2^3} - \frac{2\pi^2}{2} + \frac{4}{2^3}\right] \quad \Rightarrow \quad b_2 = \frac{1}{\pi}[-\pi^2] = -\pi$$

$$b_3 = \frac{1}{\pi}\left[-\frac{4}{3^3} + \frac{2\pi^2}{3} - \frac{4}{3^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{8}{3^3} + \frac{2\pi^2}{3}\right] = \frac{1}{\pi}\left[\frac{2\pi^2}{3} - \frac{8}{3^3}\right]$$

$$b_4 = \frac{1}{\pi}\left[-\frac{4}{4^3} - \frac{2\pi^2}{4} + \frac{4}{4^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi^2}{4}\right] = -\frac{\pi}{2}$$

$$f(x) = \left(\frac{2\pi}{1^3} - \frac{8}{1^3 \,.\, \pi}\right)\sin x - \pi \sin 2x + \left(\frac{2\pi}{3} - \frac{8}{3^3 \,.\, \pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots\ldots$$

Example 3.111. *Show that for* $-\pi < x < \pi$

$$\sin ax = \frac{2 \sin a\pi}{\pi}\left[\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} \ldots\ldots\right]$$

Solution. Let $f(x) = \sin ax$, then

$$\sin ax = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

We know that, $\quad f(-x) = \sin(-ax) = -\sin ax = -f(x)$

Therefore, given function $f(x)$ is an odd function,

$\therefore \quad a_0 = 0, a_n = 0$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ax \sin nx \, dx$$

or
$$= \frac{2}{\pi}\int_{0}^{\pi} \sin ax \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [\cos(n-a)x - \cos(n+a)x]\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1}\sin a\pi}{n-a} - \frac{(-1)^{n+1}\sin a\pi}{n+a}\right]$$

$$= \frac{(-1)^{n+1}}{\pi}\left[\frac{2n \sin a\pi}{n^2 - a^2}\right], n \neq a$$

$$= \frac{2(-1)^{n+1}}{\pi(n^2 - a^2)} n \sin a\pi$$

$$\therefore \quad \sin ax = f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(n^2 - a^2)\pi} n \sin a\pi \sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{1}{1^2 - a^2}\sin x - \frac{2}{2^2 - a^2}\sin 2x + \frac{3}{3^2 - a^2}\sin 3x \ldots\ldots\right]$$

Example 3.112. *For a function $f(x)$ defined by $f(x) = |x|$, $-\pi < x < \pi$, obtain the Fourier series. Also deduce that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Solution. Given, $\quad f(x) = |x|$

$$\therefore \quad |x| = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

then
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} |x|\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}[(0+\pi^2)+(\pi^2-0)]=0$$

$$= \frac{2\pi^2}{2\pi}=\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}[(-1)^n - 1]\frac{1}{n^2}$$

Case I: When n is even, $a_n = 0$

Case II: When n is odd, $a_n = \frac{-4}{\pi n^2}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -(x)\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-x\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\left(-\pi\frac{(-1)^n}{n}\right) + \left(-\pi\frac{(-1)^n}{n}\right)\right] = \frac{1}{\pi}[0] = 0$$

$$\Rightarrow \qquad |x| = f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \ldots\ldots\right] \qquad \ldots(1)$$

Put $x = 0$ in (1), we have

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

> **Note:** since $|x|$ is an even function, it can be expanded as $|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Example 3.113. *Obtain Fourier series for the function f(x) =* $\begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$. *Also deduce that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Solution. Try yourself.

Example 3.114. *Obtain Fourier series for the function f(x) given by* $\begin{cases} 1+\dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$.

Hence deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}\ldots\ldots=\dfrac{\pi^2}{8}$.

Solution. Try yourself.

3.7.4 Change of Interval (Change of Scale)

We have so far considered only interval of length 2π *i.e.*, $(-\pi, \pi)$ or $(0, 2\pi)$, but most of the practical problems involves use of functions with different periods say '$2l$'. In such cases, we transform, by simple change of variable, the interval of arbitrary length *i.e.* $2l$ into length 2π, and then determine fourier series.

Let the function defined in the interval $(-l, l)$ *i.e.*, $-l < x < l$, we have to change it in, $-\pi < x < \pi$, for that we put $z = \dfrac{\pi x}{l}$ or $x = \dfrac{lz}{\pi}$ $\quad\therefore\quad f(x) = f\left(\dfrac{lz}{\pi}\right) = F(z)$

When, $\quad x = -l \;\Rightarrow\; z = -\pi$

and $\quad x = l \;\Rightarrow\; z = \pi$

$F(z)$ is defined in the interval $(-\pi, \pi)$

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz$$

where,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\,dz$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\cos nz\,dz$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\sin nz\,dz$$

by inverse transformation, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin n\frac{\pi x}{l}$$

where,
$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

3.7.5 Half Range Series

Suppose we wish to find a Fourier series for the function $f(x)$ which is defined in half interval, say $0 \le x \le \pi$ or $0 \le x \le l$. This can be done by defining a function $F(x)$ in the interval $-l \le x \le l$ such that $F(x) : f(x)$ in the interval $0 \le x \le l$.

Now if we define $F(x)$ in $-l \le x \le l$, then the series $F(x)$ can be obtained. $F(x)$ in $-l < x < l$ is chosen in either of the following two ways.

1. Half range cosine series

Let $F(x)$ be an even function in $-l < x < l$ such that $F(x) = f(x)$ in $0 \le x \le l$, we have

$$a_0 = \frac{1}{l}\int_{-l}^{l} F(x)\,dx = \frac{2}{l}\int_{0}^{l} F(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} F(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}dx = 0$$

$$\Rightarrow \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

2. Half range sine series

Let $F(x)$ be an odd function in $-l \le x \le l$, then

$$a_0 = 0, a_n = 0$$

and
$$b_n = \frac{2}{l}\int_{0}^{l} F(x)\sin\frac{n\pi x}{l}\,dx$$

or
$$= \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\Rightarrow \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

OR

1. Half range sine series

If it is required to expand the function $f(x)$ as a sine series in the interval $0 < x < l$, we extend the function to $-l < x < l$ by reflecting it in the origin, so that $f(-x) = -f(x)$ for which $a_0 = a_n = 0$. Then the function $f(x)$ will be odd and therefore, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

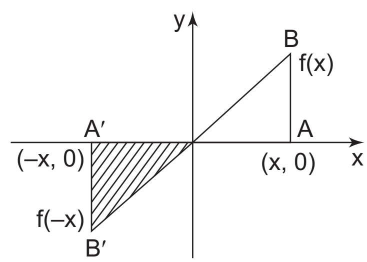


Fig. 3.5

2. Half range cosine series

If it is required to expand $f(x)$ as a cosine series in the interval $0 \le x \le l$, we extend the function to $-l \le x \le l$ by reflecting $f(x)$ in y-axis so that $f(-x) = f(x)$ for which $b_n = 0$.

Then f(x) be an even function and the fourier series will be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

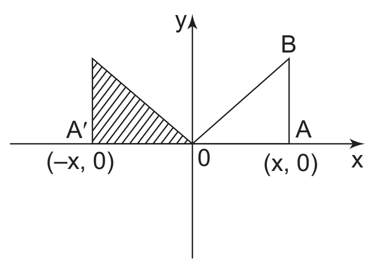


Fig. 3.6

where $$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

and $$b_n = 0$$

Note: If $l = \pi$, then $0 \le x \le \pi$ and half range sine series will be,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad a_0 = 0, a_n = 0$$

where, $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

and half range cosine series will be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

and $$a_n = \int_0^{\pi} f(x)\cos nx\,dx \text{ and } b_n = 0.$$

SOME SOLVED EXAMPLES

Example 3.115. *Expand $f(x) = e^{-x}$ as Fourier series in the interval $(-l, l)$.*

Solution. Let $$f(x) = e^{-x}$$

then, $$e^{-x} = f(x) = \frac{a_0}{2} + \Sigma a_n \cos\frac{n\pi x}{l} + \Sigma b_n \sin\frac{n\pi x}{l}$$

where, $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx \quad \Rightarrow \quad \frac{1}{l}[e^{l} - e^{-l}] = \frac{2}{l}\sinh l$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1 + \frac{n^2\pi^2}{l^2}}\left\{-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right\}\right|_{-l}^{l}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{2l(-1)^n}{l^2 + n^2\pi^2}\sinh l$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n^2\pi^2}{l^2}\right)}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \left[\frac{2n\pi(-1)^n \sinh l}{l^2 + n^2\pi^2}\right]$$

Thus,
$$e^{-x} = \frac{\sinh l}{l} + \sum \frac{2l(-1)^n}{l^2+n^2\pi^2}\sinh l \cos\frac{n\pi x}{l} + \sum \frac{2n\pi(-1)^n}{l^2+n^2\pi^2}\sinh l \sin\frac{n\pi x}{l}$$

Example 3.116. *Expand as Fourier series for the function* $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$.

Solution. Let $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}, l = 1$

then
$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi$$

$$a_n = \int_0^2 f(x)\cos\frac{n\pi x}{1}\,dx = \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[x\frac{\sin n\pi x}{n\pi} - \frac{(-\cos n\pi x)}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\frac{\sin n\pi x}{n\pi} + \left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] + \pi\left[\left(\frac{-\cos 2n\pi}{n^2\pi^2}\right) + \frac{\cos n\pi}{n^2\pi^2}\right]$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= 2\pi\left[\frac{(-1)^n - 1}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= \begin{Bmatrix} 0, & \text{when } n \text{ is even} \\ \frac{-4}{n^2\pi}, & \text{when } n \text{ is odd} \end{Bmatrix}$$

$$b_n = \int_0^2 f(x)\sin n\pi\, x\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[x\frac{(-\cos n\pi x)}{n\pi} - \frac{(-\sin n\pi x)}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\frac{(-\cos n\pi x)}{n\pi} + \frac{(-\sin n\pi x)}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\frac{-\cos n\pi}{n\pi}\right] + \pi\left[\frac{-\sin 2n\pi}{n^2\pi^2} + \frac{\cos n\pi}{n\pi}\right]$$

$$= \pi\left[-\frac{(-1)^n}{n\pi}+\frac{(-1)^n}{n\pi}\right]=0$$

$$f(x) = \frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos \pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+......\right] \qquad ...(1)$$

Put $x = 0$ in (1), we have

$$\Rightarrow \qquad f(0) = 0 = \frac{\pi}{2}-\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}......\right]$$

Thus, $\quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...... = \frac{\pi^2}{8}$

Example 3.117. *Develop f(x) in Fourier series for the interval (–2, 2) if* $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$.

Solution. Let $\quad f(x) = \frac{a_0}{2}+\sum a_n \cos\frac{n\pi x}{2}+\sum b_n \sin\frac{n\pi x}{2}$

Here, $\quad a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 0\,dx+\int_{0}^{2} 1\,dx\right] = \frac{1}{2}\times 2 = 1$

and $\quad a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$

$$= \frac{1}{2}\int_{0}^{2} \cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_0^2 = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\,.\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{0}^{2} 1\,.\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{-1}{n\pi}\left[\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{-1}{n\pi}[\cos n\pi - \cos 0] = \frac{1}{n\pi}[1-(-1)^n]$$

$$b_n = \begin{cases} \frac{1}{n\pi}[1-(-1)^n] = \frac{2}{n\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\Rightarrow \quad b_2 = 0, b_4 = 0$

$\therefore \quad f(x) = \frac{1}{2}+\frac{2}{\pi}\sin\frac{\pi x}{2}+\frac{2}{3\pi}\sin\frac{3\pi x}{2}+\frac{2}{5\pi}\sin\frac{5\pi x}{2}+......$

Example 3.118. *A sinusoidal voltage E sin wt is passing through a half wave rectifier which clips the negative portion of the wave. Develop the resulting function,*

$$u(t) = \begin{Bmatrix} 0, & \frac{-T}{2} < t < 0 \\ E \sin wt, & 0 < t < \frac{T}{2} \end{Bmatrix} \quad \text{and} \quad T = \frac{2\pi}{w} \text{ in a Fourier series.}$$

Solution. Try yourself.

Example 3.119. *Obtain sine and cosine series for f(x) = x in the interval $0 \le x \le \pi$. Hence show that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Solution. For sine series

Let us extend its interval to $-\pi \le x \le \pi$ by reflecting $f(x)$ in the origin so that $f(-x) = -f(x)$. Hence $f(x)$ odd.

$$a_0 = 0 \text{ and } a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx\, dx = \frac{2}{\pi}\left[x\frac{(-\cos nx)}{n} - \left(\frac{-\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n} - (0)\right] = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\therefore \qquad f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

For cosine series

Let us extend the interval to $-\pi \le x \le \pi$, by reflecting $f(x)$ in y-axis, such that $f(-x) = f(x)$. Hence $f(x)$ is even, therefore $b_n = 0$ and

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \frac{(-\cos nx)}{n^2}\right]_0^\pi = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$= \begin{Bmatrix} \frac{-4}{\pi n^2}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{Bmatrix}$$

$$\therefore \qquad f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} ......\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots\right]$$

$$\therefore \qquad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots = \frac{\pi^2}{8}.$$

Example 3.120. *Find the half range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right)$.*

Solution. Extend the interval of $f(x)$ to $-1 < x < 1$ by reflecting $f(x)$ in y-axis such that $f(-x) = f(x)$ and the new function is even function in $(-1, 1)$

$$b_n = 0 \text{ and } a_0 = \frac{2}{1}\int_0^1 (x-1)^2\, dx = \frac{2}{1}\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$= \frac{2}{3},$$

$$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos\frac{n\pi x}{1}\, dx$$

$$= \frac{2}{1}\left[(x-1)^2 \frac{\sin n\pi x}{n\pi} - 2(x-1)\frac{(-\cos n\pi x)}{n^2\pi^2} + 2\frac{(-\sin n\pi x)}{n^3\pi^3}\right]_0^1$$

$$= 2\left[0 + \frac{2}{n^2\pi^2}\right] = \frac{4}{n^2\pi^2}$$

Therefore half range cosine series is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\left[\sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2}\right]$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} \ldots\ldots\right] \qquad \text{[Part I Proved]}$$

Part II, do yourself.

Example 3.121. *Obtain a Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$. Also show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots. = \frac{\pi - 2}{4}$.*

Solution. For cosine series, let us extend the interval of $f(x)$ to $-\pi < x < \pi$ by reflecting $f(x)$ in y-axis such that $f(-x) = f(x)$, then $f(x)$ is even function in the interval $(-\pi, \pi)$, therefore $b_n = 0$ and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\, dx = \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}[-\pi \cos \pi] = 2,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} x(2\cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right)-1\left(\frac{-\sin(n+1)x}{(n+1)^2}-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}\right)\right], n \neq 1$$

$$= \left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n+1}}{n-1}\right]$$

$$= (-1)^{n+1}\left[\frac{1}{n-1}-\frac{1}{n+1}\right]=\frac{2(-1)^{n+1}}{n^2-1}$$

$$= \frac{2(-1)^{n+1}}{n^2-1}, n \neq 1$$

When $n = 1$, $\quad a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-1.\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= -\frac{1}{2}$$

$$x \sin x = 1-\frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1.3}-\frac{\cos 3x}{2.4}+\frac{\cos 4x}{3.5}\ldots\ldots\right] \qquad \text{[Part I Proved]}$$

Part II, do yourself.

Example 3.122. *If* $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$

Show that

i. $f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x\right)$

ii. $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x\right]$

Solution. Try yourself.

Example 3.123. *Obtain half range cosine series* $f(x) = \begin{cases} Kx, & 0 \le x \le \frac{l}{2} \\ K(l-x), & \frac{l}{2} \le x \le l \end{cases}$

Deduce the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots$

Solution. Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} Kx\,dx + \int_{l/2}^l K(l-x)\,dx\right]$$
$$= \frac{2}{l}\left[\left(K\frac{x^2}{2}\right)_0^{l/2} + K\left((lx) - \frac{x^2}{2}\right)_{l/2}^{l}\right]$$
$$= \frac{2}{l}\left[\left(K\frac{l^2}{8}\right) + K\left(\left(l^2 - \frac{l^2}{2}\right) - K\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right)\right]$$
$$= \frac{2}{l}\left[K\frac{l^2}{4}\right] = \frac{Kl}{2}$$
$$a_n = \frac{2}{l}\left[\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx\right]$$
$$= \frac{2}{l}\left[\int_0^{l/2} K\,x\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^l K(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$
$$= \frac{2K}{l}\left\{\left[x\left(\frac{\frac{\sin n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \left[(l-x)\frac{\frac{\sin n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$
$$= \frac{2K}{l}\left[\left(\frac{\frac{l}{2}}{\frac{n\pi}{l}}\sin\frac{n\pi}{2} + \frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}} - \frac{l}{\frac{n^2\pi^2}{l^2}}\right) + \left(\frac{-l^2}{n^2\pi^2}\cos n\pi - \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right)\right]$$
$$= \frac{2K}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$
$$= \frac{2K}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1$
$$a_1 = a_3 = a_5 = \ldots\ldots = 0$$

When n is even
$$a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 2\right]$$
$$a_2 = \frac{4kl}{2^2\pi^2}[-2] = \frac{-8kl}{2^2\pi^2}$$
$$a_4 = \frac{4kl}{4^2\pi^2}[\cos 2\pi - 1] = 0$$

$$a_6 = \frac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - \cos 6\pi] = \frac{-8kl}{6^2\pi^2} \text{ and so on}$$

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} \ldots\ldots\right] \qquad \text{[Part I Proved]}$$

Part II, Do yourself.

EXERCISE 3.15

1. Find Fourier expansion of $f(x) = x - x^2, -1 < x < 1$.
2. Expand as Fourier expansion of $f(x) = e^{-x}$ in $(-1, 1)$.
3. Expand $f(x) = 0, -2 < x < 0$ and $f(x) = 1, 0 < x < 2$ as Fourier series.
4. Expand $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ as Fourier series.
5. Expand $f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ as Fourier series.
6. Obtain Fourier expansion of $f(x) = |\cos x|$ in $(-\pi, \pi)$,
7. Expand $f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \end{cases}$ as Fourier sine series.
8. Prove that in, $0 < x < l$
$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left\{\frac{\cos \pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\right\}$$
Hence deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$.
9. Show that a constant function 'a' can be expanded as $\frac{4a}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right\}$ in the range, $0 < x < \pi$.
10. Show that $0 < x < \pi$, $x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2}\right)$ and hence evaluate $\sum_{n=1}^{\infty}\frac{1}{n^4}$.
11. Obtain sine series for unity in, $0 < x < \pi$ and hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.
12. Prove that in the range $(0, l)$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos(2m-1)\frac{\pi x}{l}\right]$ and hence deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$.
13. Obtain a Fourier series for $f(x) = 1 - t^2, -1 \le t \le 1$.

14. Expand as Fourier series $f(x) = \begin{cases} \dfrac{1}{4} - x, & 0 < x < \dfrac{1}{2} \\ x - \dfrac{1}{4}, & \dfrac{1}{2} < x < 1 \\ 1, & 1 < x < \dfrac{3}{2} \\ x - 1, & \dfrac{3}{2} < x < 2 \end{cases}$

15. Develop $\sin\left(\dfrac{\pi x}{l}\right)$ as half range cosine series in $0 < x < l$.

Answers

1. $x - x^2 = -\dfrac{1}{3} + \dfrac{4}{\pi^2}\left[\dfrac{\cos \pi x}{1^2} - \dfrac{\cos 2\pi x}{2^2} + \dfrac{\cos 3\pi x}{3^2} \ldots\ldots\right] + \dfrac{2}{\pi}\left[\dfrac{\sin \pi x}{1} \ldots\ldots\right]$

2. $e^{-x} = \sinh l\left[\dfrac{1}{l} - 2l\left[\dfrac{1}{l^2+\pi^2}\cos\dfrac{\pi x}{l} - \dfrac{1}{l^2 2^2 \pi^2}\sin\dfrac{2\pi x}{l}\right] + \dfrac{1}{l^2 + 3^3\pi^2}\cos\dfrac{3\pi x}{l} + \ldots..\right]$

$$-2\pi\left[\frac{1}{e^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{3^2+2^2\pi^2}\sin\frac{2\pi x}{l} \ldots\ldots\right]$$

3. $f(x) = \dfrac{1}{4} - \dfrac{2}{\pi^2}\left[\cos \pi x + \dfrac{\cos 3\pi x}{3^2} + \dfrac{\cos 5\pi x}{5^2} \ldots\ldots\right]$

4. $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi}\left[\sin\dfrac{\pi x}{2} + \dfrac{1}{3}\sin\dfrac{3\pi x}{2} + \dfrac{1}{5}\sin\dfrac{5\pi x}{2}\right]$

5. $f(x) = \dfrac{k}{2} + \dfrac{2k}{\pi}\left[\cos\dfrac{\pi t}{2} - \dfrac{1}{3}\cos\dfrac{3\pi t}{2} + \dfrac{1}{5}\cos\dfrac{5\pi t}{2}\right]$

6. $|\cos x| = \dfrac{2}{\pi} + \dfrac{4}{\pi}\left[\dfrac{1}{3}\cos 2x - \dfrac{1}{15}\cos 4x + \ldots\ldots\right]$

7. $f(x) = \left(\dfrac{1}{\pi} - \dfrac{4}{\pi^2}\right)\sin \pi x + \left(\dfrac{1}{3\pi} - \dfrac{4}{3^2\pi^2}\right)\sin 3\pi x - \left(\dfrac{1}{5\pi} - \dfrac{4}{5^2\pi^2}\right)\sin 5\pi x - 1$

11. $1 = \dfrac{4}{\pi}\left[\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right]$

13. $f(x) = \dfrac{7}{16} + \dfrac{1}{\pi}\left(\dfrac{1}{\pi} + \dfrac{1}{4}\right)\cos \pi x - \dfrac{3}{4}\left(\dfrac{1}{\pi} + \dfrac{1}{2}\right)\sin \pi x$

14. $1 - t^2 = \dfrac{2}{3} + \dfrac{4}{\pi^2}\left[\cos \pi t - \dfrac{\cos 2\pi t}{2^2} + \dfrac{\cos 3\pi t}{3^2} + \ldots\ldots\right]$

15. $\sin\left(\dfrac{\pi x}{l}\right) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\left[\dfrac{\frac{\cos 2\pi x}{l}}{1.3} + \dfrac{\frac{\cos 4\pi x}{l}}{3.5} + \dfrac{\frac{\cos 6\pi x}{l}}{5.7} + \ldots\ldots\right]$

14. $1-t^2 = \frac{2}{3}+\frac{4}{\pi^2}\left[\cos\pi t - \frac{\cos 2\pi t}{2^2}+\frac{\cos 3\pi t}{3^2}+\ldots\ldots\right]$

15. $\sin\left(\frac{\pi x}{l}\right)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\frac{\cos 2\pi x}{l}}{1.3}+\frac{\frac{\cos 4\pi x}{l}}{3.5}+\frac{\frac{\cos 6\pi x}{l}}{5.7}+\ldots\ldots\right]$

3.7.6 Parseval's Theorem on Fourier Series

Statement

If the fourier series of $f(x)$ over an interval $c < x < c + 2l$ is given as

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

then

$$= \frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0^{\,2}}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2}+b_n^{\,2})$$

Proof: Now we have the interval $c < x < c + 2l$,

$$\therefore \quad f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(i)$$

where

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

and

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Multiplying both sides of (i) by $f(x)$, we have

$$[f(x)]^2 = \frac{a_0}{2}f(x)+\sum_{n=1}^{\infty} a_n\, f(x)\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n\, f(x)\sin\frac{n\pi x}{l}\,dx$$

Integrating both sides w.r.t. x, between the limit c to $c + 2l$, we have

$$\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0}{2}\int_c^{c+2l} f(x)\,dx+\sum_{n=1}^{\infty} a_n\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx+\sum_{n=1}^{\infty} b_n\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx \qquad \ldots(1)$$

$$= \frac{a_0}{2}a_0 l+\sum_{n=1}^{\infty} a_n\,(la_n)+\sum_{n=1}^{\infty} b_n\,(lb_n)$$

$$= \frac{a_0^{\,2}}{2}l+\sum_{n=1}^{\infty} la_n^{\,2}+\sum_{n=1}^{\infty} lb_n^{\,2}$$

Case I: If $c = 0$, then (1) becomes,

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2\,dx = \frac{a_0^{\,2}}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2}+b_n^{\,2})$$

Case II. If $c = -l$, the interval becomes $-l < x < l$, then

i. If $f(x)$ is even in $(-l, l)$, then $\frac{2}{l}\int_0^l [f(x)]^2\, dx = \frac{a_0^{\,2}}{2} + \sum_1^{\infty} a_n^{\,2}$

ii. If $f(x)$ is odd in $(-l, l)$, then $\frac{2}{l}\int_0^l [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^{\,2}$

SOME SOLVED EXAMPLES

Example 3.124. *Find the fourier series for unity in $0 < x < \pi$ and hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.*

Solution. Since we have half range sine series for half the interval for $f(x) = 1$ in $0 < x < \pi$,

$$\therefore \qquad a_0 = 0,\ a_n = 0 \text{ and } f(x) = \sum_{n=1}^{\infty} b_n \sin nx\, dx$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-2}{\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n}\right] = \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{4}{\pi n} & \text{when } n \text{ is odd} \end{cases}$$

Now from Parseval's theorem on Fourier series,

$$\int_c^{c+2l} [f(x)]^2\, dx = 2l\left[\frac{a_0^{\,2}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})\right]$$

$$l = 0 \text{ and } 2l = \pi \quad \Rightarrow \quad \int_0^{\pi} (1)^2\, dx = \frac{\pi}{2}\left[\frac{16}{\pi^2 1^2} + \frac{16}{\pi^2 3^3} + \frac{16}{\pi^2 5^2}\ldots\ldots\right]$$

or,
$$\pi = \frac{8}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\right]$$

Thus,
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots$$

Example 3.125. *Prove that, in $0 < x < l$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l}\ldots\ldots\right]$*

hence deduce that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots$

Solution. Here $\qquad f(x) = x$

We have to expand $f(x)$ in half of interval $0 < x < l$ as cosine series for which

$$x = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad ...(1)$$

where, $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{2}{l}\left[\frac{l^2}{2}\right] = l$$

and $$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\frac{\dfrac{x\sin n\pi x}{l}}{\dfrac{n\pi}{l}} - \frac{-\cos\dfrac{n\pi x}{l}}{\left(\dfrac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{l}{n\pi}\right)^2[(-1)^n - 1]\right]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1] = 0, \text{ if } n \text{ is even}$$

$$= \frac{-4l}{n^2\pi^2}, \qquad \text{if } n \text{ is odd}$$

$\therefore$ From (1), $$x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + ...\right]$$

Now $$\int_0^l [f(x)]^2\,dx = \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 ......\right]$$

$$\left[\frac{x^3}{3}\right]_0^l = \frac{l}{2}\left\{\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + ...\right\}$$

$$= \frac{l}{2}\left[\frac{l^2}{2} + \frac{16l^2}{1^4\pi^4} + \frac{16l^2}{3^4\pi^4} + \frac{16l^2}{5^4\pi^4} ......\right]$$

$\Rightarrow$ $$\frac{l^3}{3} = \frac{l^3}{2}\left[\frac{1}{2} + \frac{16}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4}\right\}\right]$$

or $$\frac{1}{3} - \frac{1}{4} = \frac{16}{2\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} ......\right\}$$

$\Rightarrow$ $$\frac{\pi^4}{12} = 8\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} ......\right]$$

$\Rightarrow$ $$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} ......$$

EXERCISE 3.16

1. Find Fourier series of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that $\dfrac{\pi^4}{90} = 1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \ldots\ldots$

2. If $f(x) = \sum b_n \sin \dfrac{n\pi x}{l}$ in $(0, l)$, then show that $\int_0^l [f(x)]^2\, dx = \dfrac{l}{2} \sum_{n=1}^{\infty} b_n^{\,2}$.

3. If $f(x) = \dfrac{a_0}{2} + \sum a_n \cos \dfrac{n\pi x}{l}$ is a half range cosine series of $f(x)$ of period $2l$ in $(0, l)$, then show that $f(x) = \dfrac{a_0^{\,2}}{2} + \sum a_n^{\,2}$, using this result to evaluate $1^{-4} + 3^{-4} + 5^{-4} + \ldots\ldots$ in the half range cosines series for the function defined in $(0, 2)$ by $f(x) = \begin{Bmatrix} \pi x, & 0 < x < 1 \\ \pi(2 - x), & 1 < x < 2 \end{Bmatrix}$.

Answers

1. $x^2 = \dfrac{\pi^2}{3} + \sum_{n=1}^{\infty} \dfrac{4(-1)^n}{n^2} \cos nx$

VIDEO REFERENCES

Introduction to Fourier Series

Fourier Series (1)

REAL LIFE EXAMPLE

1. Fibonacci numbers are simply obtained using the following simple formula for $n > 1$.

$$F_n = F_{n-1} + F_{n-2}$$

This gives us the following sequence that goes to infinity:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,

The beauty of this sequence is that it is related to nature.

For example, it appears in the flowering of the artichoke, some flower petals such as Daisies, honeybees, etc.

Think: Does it even occur in the galaxy spirals?

2. There is one more very interesting observation that the dimensions of the Earth and Moon are in Phi relationship, forming a Triangle based on 1.618.

Think: What is Phi, and what is this 1.618?

1. If we take any two successive numbers in the sequence, their ratio

(X_n/X_{n-1}) gets closer to 1.618 which is what we call the golden ratio:

$$3/2 = 1.5$$
$$13/8 = 1.666$$
$$55/34 = 1.61764$$
$$233/144 = 1.61805$$
....
$$317{,}811 \,/\, 196{,}418 = \mathbf{1.61803}$$

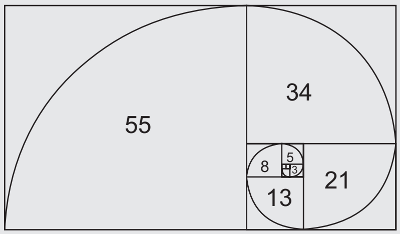


Fig. 3.7

Going to infinity, the ratios get closer to **1.618**, also known as Phi (φ).

- It appears in **Nature** (as previously mentioned). Some tree branches are an example. The main trunk will grow until it produces a branch thus creating two new starting points.

The pattern is similar to the Fibonacci pattern shown in given figure:

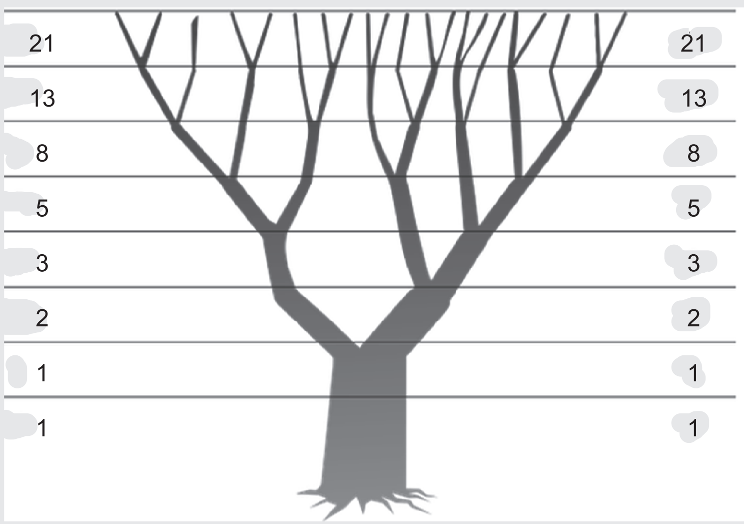


Fig. 3.8

INTERESTING FACTS

It is believed to represent Beauty and even though this belief is not proven, it remains interesting to know how our minds define beauty. For example, the face. Now, the following is probably not the most accurate research made but Dr. Schmid has a 10 scale ratio with 10 as the highest (most beautiful person) with most people scoring between 4 and 6. The beauty metric is first measured by the length and the width of the face then divides by the width. The optimal result is 1.618. Meaning a beautiful person's face is 1.618 longer than it's width. Later on, other ratios are calculated such as the bottom of the nose to the bottom of the chin. Finally, symmetry tests are performed to check out more beauty metrics. Dr. Schmid says that the length of the ear should be equal to the length of the nose on a perfect face, among other characteristics.

- It is believed that the ratio of our arm to our forehand is equal to Golden Ratio.
- It is present in Geometry. Many buildings and artworks have the Golden Ratio in them, An example would be the Parthenon in Greece.

The pantagram has the golden ratio embedded inside it.

Fig. 3.9

- Fibonacci Day is November 23rd, as it has the digits "1, 1, 2, 3" which is part of the sequence. So next Nov 23 let everyone know.

SUBJECTIVE SOLVED QUESTIONS
(HOTS)

Example 1. *Prove that if a power series $\Sigma a_n x^n$ converges for $x = b$, then it converges absolutely for all x such that $|x| < |b|$.*

Solution. Since $\Sigma a_n b^n$ converges,

$$\therefore \quad \lim_{n \to \infty} |a_n b^n| = 0$$

Since, a convergent sequence is bounded, there exists an M such that $|a_n b^n| \le M \ \forall \ n$

Let $$\left|\frac{x}{b}\right| = r < 1$$

Then $$|a_n x^n| = |a_n b^n| \, . \left|\frac{x^n}{b^n}\right| < Mr^n$$

Therefore, by comparison with the convergent geometric series ΣMr^n, $\Sigma|a_n x^n|$ is convergent.

Example 2. *Given that $n! \ge 2^{n-1}$ for all $n \ge 1$, show that the sequence $<a_n>$ whose n^{th} term is $a_n = \sum_{n=0}^{k} \frac{1}{n!}$ is bounded above by 3. Explain why you can deduce that it converges.*

Solution. Given that $n! \ge 2^{n-1}$ for $n \ge 1$ and thus

$$\frac{1}{n!} \le \frac{1}{2^{n-1}} \text{ for } n \ge 1.$$

Thus,
$$a_n = 1 + \sum_{n=1}^{k} \frac{1}{n!}$$
$$\le 1 + \sum_{n=1}^{k} \frac{1}{2^{n-1}}$$
$$= 1 + \frac{1-\frac{1}{2^k}}{1-\frac{1}{2}} \qquad \text{(by summing the geometric series)}$$
$$= 1 + 2\left(1-\frac{1}{2^k}\right) < 3$$

Since $a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$ the sequence is strictly increasing and as it is bounded above. By monotonic convergence theorem, it converges.

Example 3. a. *Evaluate the integral* $\int_0^{\pi/6} sin^2 x\, dx$ *by first finding the Maclaurin approximation to the integrand with 3 terms.*

b. *Evaluate the integral exactly and compare.*

Solution. a. The Maclaurin series for $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$ ......

Hence
$$\int_0^{\pi/6} \sin^2 x dx = \int_0^{\pi/6}\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \ldots\right)dx = \left(\frac{x^3}{3} - \frac{x^5}{15} + \frac{2x^7}{315} - \ldots\right)\Bigg|_0^{\pi/6} = 0.0452941$$

b. The exact integral is
$$\int_0^{\pi/6} \sin^2 x dx = \int_0^{\pi/6} \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4}\Bigg|_0^{\pi/6} = \frac{\pi}{12} - \frac{\sin(\pi/3)}{4}$$
$$= \frac{\pi}{12} - \frac{\sin(\pi/3)}{4} = \frac{2\pi - 3\sqrt{3}}{24} = 0.045293$$

Example 4. *The relationship between the wavelength, L, the wave period, T, and the water depth, d, for a surface wave in water is given by:*
$$L = \frac{gT^2}{2\pi} \tan h\left(\frac{2\pi d}{L}\right)$$

In a particular case the wave period was 105 and the water depth was 6.1 m. Taking the acceleration due to gravity, g as 9.81 ms^{-2}. Determine the wave length by using series expansion for tan hx.

Solution. Substituting for the wave period, water depth and g, we get
$$L = \frac{9.81\times 10^2}{2\pi} \tan h\left(\frac{2\pi\times 6.1}{L}\right) = \frac{490.5}{\pi} \tan h\left(\frac{12.2\,\pi}{L}\right)$$

Series expansion of tan hx is given by

$$\tan h\,x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$$

Using the series expansion of tan $h\,x$, we can approximate the equation as

$$L = \frac{490.5}{\pi}\left(\frac{12.2\pi}{L} - \frac{1}{3}\left(\frac{12.2\pi}{L}\right)^3 + \ldots\right)$$

Multiplying through by πL^3, the equation becomes

$$\pi L^4 = 490.5 \times 12.2\,\pi\,L^2 - \frac{490.5}{3}\;(12.2\pi)^3$$

The equation can be re-written as

$L^4 - 5984.1\,L^2 + 2930198 = 0$

Solving this as a quadratic in L^2, we get $L = 74$ m

Example 5. a. *Find the Fourier coefficients corresponding to the function*

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \quad \textit{Period} = 10$$

b. *Write the corresponding Fourier series.*

c. *How should $f(x)$ be defined at $x = -5$, $x = 0$, and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \le x \le 5$?*

Solution. The graph of $f(x)$ is shown in Fig.

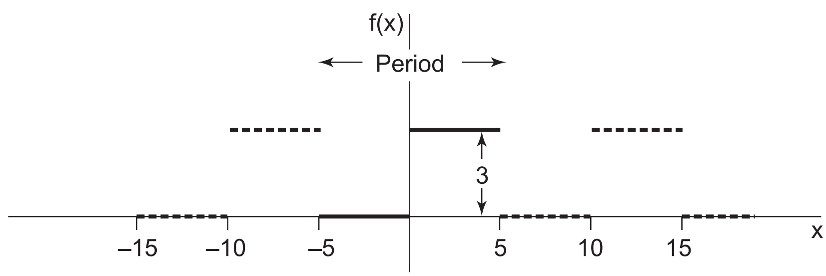


Fig. 3.10

a. Period = $2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \text{ if } n \ne 0$$

If $n = 0$, $a_n = a_0 = \dfrac{3}{5}\displaystyle\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_{0}^{5}(3)\sin\frac{n\pi x}{5}\,dx\right\}\ \frac{3}{5}\int_{0}^{5}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_{0}^{5} = \frac{3(1-\cos n\pi)}{n\pi}$$

b. The corresponding Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \ldots\right)$$

c. Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0, and 5, which are points of discontinuity, the series converges to $(3 + 0)/2 = 3/2$ as seen from the graph. If we redefine $f(x)$ as follows

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

then the series will converge to $f(x)$ for $-5 \le x \le 5$.

Example 6. *Find a Fourier series for $f(x) = \cos ax$, $-\pi \le x \le \pi$, where $a \ne 0, \pm 1, \pm 2, \pm 3 \ldots$*

Solution. Try yourself.

SUMMARY

1. An ordered set of numbers such as $a_1, a_2, \ldots, a_n, \ldots$ is called a sequence.
2. Limit of a convergent sequence is always unique.
3. A bounded sequence either converges or oscillates finitely.
4. Infinite series is denoted by $\sum_{n=1}^{\infty} u_n$, and to every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.
5. An infinite series is said to converge, diverge or oscillate according as its sequence of partial sums $\{S_n\}$ converges, diverges or oscillates.
6. **Cauchy's General Principle of Convergence:** A necessary and sufficient condition for series $\sum_{n=1}^{\infty} u_n$ to converge is that for each $\varepsilon > 0$, there exists a positive integer m such that
$$|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon,\ n \ge m$$
7. An infinite series all of whose terms are positive is called a positive term series.
8. The infinite series $\sum \frac{1}{n^p}$ is known as harmonic series or simply 'p-series'. The series converges if

$p > 1$ and diverges if $p \leq 1$.

9. **D' Alembert Ratio test:** If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$,
   i. for $l > 1$, series is convergent
   ii. for $l < 1$, the series is divergent
   iii. for $l = 1$, no conclusion (test fails).

10. **Cauchy's Root test:** If $\sum_{n=1}^{\infty} a_n$ is a positive term series and $\lim_{n \to \infty} (a_n)^{1/n} = l$, then
    i. Σa_n is convergent if $l < 1$
    ii. Σa_n is divergent if $l > 1$
    If $l = 1$, test fails, then in that case, to find convergence or divergence, apply another test.

11. **Raabe's test:** If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms such that $\lim_{n \to \infty} n\left[\frac{a_n}{a_{n+1}} - 1\right] = l$, then the series
    i. convergent if $l > 1$
    ii. divergent if $l < 1$
    iii. test fails if $l = 1$.

12. **Logarithmic test:** If $\sum_{n=1}^{\infty} a_n$ is a series of positive term such that $\lim_{n \to \infty} n \log \frac{a_n}{a_{n+1}} = l$, then the series will be
    i. convergent if $l > 1$
    ii. divergent if $l < 1$
    iii. test fails if $l = 1$.

13. **Gauss test:** If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms such that $\frac{a_n}{a_{n+1}}$ can be expanded in the form,
    $$\frac{a_n}{a_{n+1}} = 1 + \frac{\lambda}{n} + 0\left(\frac{1}{n^2}\right)$$
    Then Σa_n converges if $\lambda > 1$ and diverges if $\lambda \leq 1$.
14. **Alternating series:** A series of the form $a_1 - a_2 + a_3 - a_4 + \ldots$, where $a_n > 0 \ \forall\ n$, is called an alternating series and is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$.
15. **Leibnitz's test for convergence of alternating series:** The alternating series
    $$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots \qquad (a_n > 0 \text{ for all } n)$$
    is convergent if
    i. $a_{n+1} < a_n \ \forall\ n$
    ii. $\lim_{n \to \infty} a_n = 0$
16. **Absolutely convergent series:** A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.
17. **Conditionally convergent series:** A series is said to be conditionally convergent if it is convergent

but does not converge absolutely.

18. **Taylor's infinite series:** $f(x)$ is ascending powers of $(x - a)$ is given by,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots$$

19. Maclaurin's infinite series is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

20. Fourier series is given by $\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$

21. **Dirichlet's condition for fourier series:** If a function $f(x)$ is defined in the interval $c \le x \le c + 2\pi$ and is

i. finite, single valued in the interval

ii. has a finite number of singularities in the interval

iii. is periodic

iv. has infinite number of maxima and minima in the interval.

22. Even and odd functions

i. If $f(-x) = f(x)$, then the function $f(x)$ is an even function.

ii. If $f(-x) = -f(x)$, then the function $f(x)$ is an odd function.

23. **Half range sine series:** Half range sine series for a function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, -l < x < l$$

24. **Half range cosine series:** Half range cosine series for a function $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, -l < x < l$$

OBJECTIVE QUESTIONS

1. Let $a_n = \sin n \left(\frac{\pi}{4}\right)$, for the sequence $a_1, a_2, \ldots$, the supremum is

a. 0 and it is attained
b. 0 and it is not attained
c. 1 and it is attained
d. 1 and it is not attained

2. The sequence $\{x_n\}$, where $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2^n}$ is

a. increasing but not bounded
b. increasing and bounded
c. decreasing and bounded
d. decreasing but not bounded

3. The sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}$, converges to

a. $\frac{1+\sqrt{33}}{2}$
b. $\frac{1+\sqrt{32}}{2}$
c. $\frac{1+\sqrt{30}}{2}$
d. $\frac{1+\sqrt{29}}{2}$

4. Which of the following statements is true?

a. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim_{x\to\infty} \frac{\log x}{x} = \infty$

b. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = \infty$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$

c. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$

d. $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ but $\lim_{x\to\infty} \frac{\log x}{x}$ does not exist

5. Consider the statements

A. $\lim_{n\to\infty} \left(\frac{(3n)!}{(n!)^3}\right)^{1/n} = 27$

B. $\lim_{n\to\infty} \left(\frac{n^n}{(n+1)(n+2)\dots(n+n)}\right)^{1/n} = \frac{e}{4}$

a. (A) is true but (B) is false

b. (A) is false but (B) is true

c. (A) and (B) both are true

d. Neither (A) nor (B) is true

6. $\lim_{n\to\infty} \frac{e + e^{1/2} + e^{1/3} + \dots + e^{1/n}}{n}$ is equal to

a. 0

b. 1

c. e

d. None of these

7. $\lim_{n\to\infty} \frac{1}{n}[1 + 2^{1/2} + 3^{1/3} + \dots + n^{1/n}]$ equal to

a. 0

b. 1

c. e

d. None of these

8. $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \dots + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\right)$ equals

a. $\sqrt{2}$

b. $\frac{1}{\sqrt{2}}$

c. $\sqrt{2}+1$

d. $\frac{1}{\sqrt{2}+1}$

9. Consider the statements

A. The series $\sum \sin\frac{1}{n}$ is divergent

B. The series $\frac{1}{3^2} \cdot \frac{2}{4^2} + \frac{3.4}{5^2.6^2} + \frac{5.6}{7^2.8^2}$ is convergent.

Then

a. Both the statements (A) and (B) are true

b. (A) is true but (B) is false.

c. (A) is false but (B) is true.

d. Neither (A) nor (B) is true.

10. For which real number 't' does the infinite series $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n^t}$ converge

a. $t > \frac{1}{3}$

b. $t > 1/2$

c. $t > 1$

d. $t > 3/2$

11. If $u_n = \sqrt{n+1} - \sqrt{n}$, $v_n = \sqrt{n^4+1} - n^2$, then

a. $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} v_n$ diverges

b. $\sum_{n=1}^{\infty} u_n$ diverges but $\sum_{n=1}^{\infty} v_n$ converges

c. $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ both converges

d. $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ both diverges

12. If $b_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\ \dfrac{1}{n}, & \text{if } n \text{ is even} \end{cases}$ then,

a. both $\langle b_n \rangle$ and Σb_n are convergent
b. both $\langle b_n \rangle$ and Σb_n are divergent
c. $\langle b_n \rangle$ convergent but Σb_n not
d. Σb_n convergent but $\langle b_n \rangle$ not

13. Using the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ equal to

a. $\frac{\pi^2}{12}$
b. $\frac{\pi^2}{12} - 1$
c. $\frac{\pi^2}{8}$
d. $\frac{\pi^2}{8} - 1$

14. Which of the following is convergent?

a. $\sum_{n=1}^{\infty} n^2 2^{-n}$
b. $\sum_{n=1}^{\infty} n^{-2} 2^{n}$
c. $\sum_{n=1}^{\infty} \frac{1}{n \log n}$
d. $\sum_{n=1}^{\infty} \frac{1}{n \log\left(1 + \frac{1}{n}\right)}$

15. Which of the following series is absolutely convergent?

a. $\sum \frac{(-1)^n}{n}$
b. $\sum \frac{1}{\sqrt{n}}$
c. $\sum \frac{1}{\log(n+1)}$
d. $\sum \frac{(-1)^n}{n^{3/2}}$

16. Using the fact that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ equal to

a. $1 - 2 \log 2$
b. $1 + \log 2$
c. $(\log 2)^2$
d. $-(\log 2)^2$

17. Consider the power series $f(x) = \sum_{n=2}^{\infty} \log(n)\, x^n$, then the radius of convergence of the series $f(x)$ is,

a. 0
b. 1
c. 3
d. ∞

18. The necessary condition for the Maclaurin's expansion to be true for function $f(x)$ is

a. $f(x)$ should be continuous
b. $f(x)$ should be differentiable
c. $f(x)$ should exists at every point
d. $f(x)$ should be continuous and differentiable

19. Find the expansion of $e^x \sin x$?

a. $1 + x^2 - \frac{x^4}{3} + \frac{x^6}{120}$ ......
b. $1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \ldots\ldots$
c. $x + \frac{x^3}{3} + \frac{x^5}{120} + \ldots\ldots$
d. $x + \frac{x^3}{3} - \frac{x^5}{120} + \ldots\ldots$

20. Given $f(x) = \ln(\cos(x))$, calculate the value of $\ln(\cos(\pi/2))$.

a. –1.741
b. 1.741
c. 1.563
d. –1.563

21. Let f be a function with second derivative $f''(x) = \sqrt{1+3x}$. The co-efficient of x^3 in the Taylor's series for f about $x = 0$ is

a. 1/12
b. 1/6
c. 1/4
d. 1/2

22. Let $P(x) = 3 - 3x^2 + 6x^4$ be the fourth degree Taylor polynomial for f about $x = 0$. What is the value of $f^{(4)}(0)$?

a. 0 b. 1/4 c. 144 d. 164

23. What is the co-efficient of x^2 in the Taylor's series $\sin^2 x$ about $x = 0$?

a. –2 b. –1 c. 0 d. 1

24. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ is the Taylor's series, for which of the following function?

a. $\sin x$ b. $\cos x$ c. e^x d. e^{-x}

25. Which of the following is not Dirichlet's condition for the fourier series expansion?

a. $f(x)$ is periodic, single valued, finite
b. $f(x)$ has finite number of discontinuities in only one period
c. $f(x)$ has finite number of maxima and minima
d. $f(x)$ is single valued and may be finite

26. Find the value of $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots\ldots$ by finding half range Fourier cosine series of the function $f(x) = x$ in $0 < x < 1$.

a. $\frac{\pi^4}{12}$ b. $\frac{\pi^4}{48}$ c. $\frac{\pi^4}{24}$ d. $\frac{\pi^4}{96}$

27. Find b_n in the expansion of half range sine series of the function x^2 in the interval 0 to 3.

a. $-18\frac{\cos(n\pi)}{n\pi}$ b. $18\frac{\cos(n\pi)}{n\pi}$ c. $-18\frac{\cos(n\pi/2)}{n\pi}$ d. $18\frac{\cos(n\pi/2)}{n\pi}$

28. Find the sum of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ using Fourier series expansion if $f(x) = a$ when $[0, \pi]$ and $2\pi - x$ when $[\pi, 2\pi]$

a. $\pi^2/8$ b. $\pi^2/4$ c. $\pi^2/16$ d. $\pi^2/2$

29. Find the half range sine series for the function $f(x) = x$, when $0 < x < \frac{\pi}{2}$ and $(\pi - x)$ when $\frac{\pi}{2} < x < \pi$

a. $\frac{8}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \frac{\sin(7x)}{7^2} + \ldots\right]$ b. $\frac{4}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \frac{\sin 7x}{7^2} + \ldots\right]$

c. $\frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \frac{\sin 7x}{7^2} + \ldots\right]$ d. $\frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \ldots\right]$

30. The series $\frac{1}{2} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots$ is

a. convergent if $x^2 \le 1$; divergent if $x^2 > 1$ b. convergent if $x^2 > 1$; divergent if $x^2 < 1$
c. convergent if $x^2 < 1$; divergent if $x^2 \ge 1$ d. convergent if $x \le 1$; divergent if $x > 1$

Answers

1. c	**2.** d	**3.** d	**4.** c
5. c	**6.** b	**7.** b	**8.** b
9. a	**10.** b	**11.** b	**12.** c
13. d	**14.** a	**15.** d	**16.** a
17. b	**18.** d	**19.** b	**20.** a
21. c	**22.** c	**23.** d	**24.** d
25. a	**26.** d	**27.** a	**28.** b
29. d	**30.** a		

SUBJECTIVE UNSOLVED QUESTIONS
(HOTS)

1. If $\Sigma a_n x^n$ has a radius of convergence r_1 and if $\Sigma b_n x^n$ has a radius of convergence $r_2 > r_1$, what is the radius of convergence of the sum $\Sigma(a_n + b_n)\, x^n$?

2. Determine the radius of convergence and interval of convergence for the following power series.
$$\sum_{n=1}^{\infty} \frac{2^n}{n} (4x-8)^n.$$

3. Find the Maclaurin series for the function $f(x) = \ln(1 + x)$ and hence evaluate $\dfrac{\ln(1+x)}{x}$ also.

4. Given that $k^k \geq 2^k$ for all $k \geq 2$, show that the sequence $\{x_n\}$ whose nth term is $x_n = \sum_{k=1}^{n} \dfrac{1}{k^k}$ is bounded above by 3/2.

5. If $a_1 > b_1 > 0$ and a_n, b_n are defined as $a_n = \dfrac{a_{n-1}^2 + b_{n-1}^2}{a_{n-1} + b_{n-1}}$, $b_n = \dfrac{a_{n-1} + b_{n-1}}{2}$ for $n \geq 2$ prove that the sequence $<a_n>$ and $<b_n>$ are monotonic, one increasing and the other decreasing and that they tend to the same limit.

6. Prove a. $$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

b. $$\int_{-L}^{L} \sin\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = 0$$

where m and n can assume any of the values 1, 2, 3, ...

7. Obtain Fourier series for $f(x)$ of period $2l$ and defined as follows
$$f(x) = \begin{cases} l - x & 0 < x < l \\ 0 & l \leq x < 2l \end{cases}$$

Hence deduce that $-\dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots = \dfrac{\pi}{4}$ and $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots = \dfrac{\pi^2}{8}$

8. a. Show that for $-\pi < x < \pi$.

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$$

b. By integrating the result of (a), show that for $-\pi \le x \le \pi$.

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \ldots\right)$$

c. By integrating the result of (b), show that for $-\pi \le x \le \pi$.

$$x(\pi - x)(\pi + x) = 12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \ldots\right)$$

9. a. Obtain the Taylor series expansion of $\frac{1}{1-x}$ about $x = 2$.

b. Show that the series obtained in the result of (a) is convergent if $|x - 2| < 1$.

Answers

1. r_1 **2.** $R = \frac{1}{8}$ and $\frac{15}{8} \le x < \frac{17}{8}$

3. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ and $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\ldots$

7. $f(x) = \frac{l}{4} + \sum_{n=\text{odd}}^{\infty} \frac{2l}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\frac{n\pi x}{l}$

9. a. $-1 + (x-2) - (x-2)^2 + (x-2)^3 + \ldots + (-1)^{n+1}(x-2)^n + \ldots$

PROJECT/ACTIVITIES/PRACTICAL

PROJECT

1. Sketch the Graph for each of the following functions:

a. $f(x) = |x - 1| + |x + 1|$, in the interval $(-5, 5)$

b. $f(x) = \begin{cases} \sin x, 0 \le x \le \pi \\ 0, \pi \le x \le 2\pi \end{cases}$ Period $= 2\pi$

c. $f(x) = \begin{cases} 0, & 0 \le x < 2 \\ 1, & 2 \le x < 4 \\ 0, & 4 \le x < 6 \end{cases}$ Period $= 6$

ACTIVITY

1. Can you add numbers up to the infinity? (**Hint:** Answer is not what you imagine)

PRACTICAL

1. Using MATLAB, write a code to expand $\sin(x)$.

2. Plot a graph of Fibonacci sequence using MATLAB.

KNOW MORE

1. Let $<a_n>$ be a sequence of real number such that $\lim\limits_{n\to\infty} (a_n - a_{n-1}) = a$, then $\lim\limits_{n\to\infty} \frac{a_n}{n}$

a. converges to 1
b. diverges
c. converges to 0
d. converges to a

2. Given that $<a_n>$ be a sequence of real numbers satisfying $\sum\limits_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$. Then the series $\sum\limits_{n=0}^{\infty} a_n x^n$, $x \in \mathbb{R}$ is convergent,

a. nowhere on $\mathbb{R}$
b. everywhere on $\mathbb{R}$
c. on some set containing $(-1, 1)$
d. only on $(-1, 1)$

3. Let $S_n = \sum\limits_{k=1}^{n} \frac{1}{k}$, which of the following is true?

a. $S_2 n \geq \frac{n}{2}$ for every $n \geq 1$
b. S_n is a bounded sequence
c. $| S_2 n - S_{2^{n-1}} | \to 0$ as $n \to \infty$
d. $\frac{S_n}{n} \to 1$ as $n \to \infty$

4. The fourier series expansion of the periodic function

$$f(x) = \begin{cases} \frac{1}{2} + x, & -\frac{1}{2} < x \leq 0 \\ \frac{1}{2} - x, & 0 < x < 1/2 \end{cases}$$

when $f(x + 1) = f(x)$, is given by

a. $\frac{1}{4} + \frac{2}{\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \cos 2n\pi x$
b. $\frac{1}{4} + \frac{2}{\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos 2\,(2n-1)\,\pi x$
c. $\frac{1}{2} + \frac{1}{2\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \cos 4n\pi x$
d. $\frac{1}{2} + \frac{1}{\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos 2\,(2n-1)\,\pi x$

5. The Fourier series expansion of the function of $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi, \\ -\pi/2, & x = 0 \end{cases}$ when $f(x + 2\pi) = f(x)$ is given by

a. $-\frac{\pi}{4} + \sum\limits_{n=1}^{\infty} \left\{ \left(\frac{-1 + (-1)^n}{n^2 \pi} \right) \cos nx + \left(\frac{1 - 2(-1)^n}{n} \right) \sin nx \right\}$

b. $-\frac{\pi}{4} + \sum\limits_{n=1}^{\infty} \left\{ \left(\frac{1 - (-1)^n}{n^2 \pi} \right) \cos nx + \left(\frac{1 + 2(-1)^n}{n} \right) \sin nx \right\}$

c. $-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left\{\left(\frac{-1+(-1)^n}{n\pi}\right)\cos nx+\left(\frac{1+2(-1)^n}{n}\right)\sin nx\right\}$

d. $-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left\{\left(\frac{1-(-1)^n}{n\pi}\right)\cos nx+\left(\frac{1-2(-1)^n}{n}\right)\sin nx\right\}$

6. Which of the following functions grows fastest?

a. $x^{1,000,000,000,000}$ b. x^x

c. $x!$ d. $9,999,999,999,999,999,999^x$

7. The explicit formula for the geometric sequence 3, 15, 75, 375 ... is

a. $2\times 6!\times 3^{n-1}$ b. $3\times 5^{n-1}$

c. $3\times 8^{n-1}$ d. $7\times 4^{n-1}$

8. What is the radius of convergence and interval of convergence for the power series,

$$\sum_{n=0}^{\infty} m!(2x-1)^m ?$$

a. 3, 12 b. 1, 0.87 c. 2, 5.4 d. 0, 1/2

Answers

1. c **2.** c **3.** a **4.** b

5. a **6.** b **7.** b **8.** d

REFERENCES/SUGGESTED READINGS

1. Bary, N.K.(1964). A Treatise on Trigonometric Series, Pergamon Press.
2. Campbell, G.A. and R.M. Foster (1948).Fourier Integrals for practical Applications. D. Van Nostrand Company, Inc., Princeton, N.J.
3. Carslaw, H.S. (1921). Theory of Fourier Series and Integrals. St. Martin Press, Inc., New York.
4. Churchill, R.V. and J.W. Brown (1987). Fourier series and Boundary Value Problems, McGraw Hill, New York.
5. Courant R.(1988). Differential and Integral Calculus, Wiley, New York.
6. Dass, H.K. Advanced Engineering Mathematics, S. Chand Publications.
7. Garg, Reena (2019). Engineering Mathematics-I, 2nd Edition, Khanna Book Publishing Co. (P) Ltd.
8. Grewal, B.S. Higher Engineering Mathematics, Khanna Publishers.
9. Jain, R.K.; Iyengar, S.R.K. Advanced Engineering Mathematics, 2nd Edition, Narosa.
10. Knoop, K. (1987). Theory and Applications of Infinite Series. Blackie & Sons Ltd., Glasgow.
11. Philip, Franklin (1940). A Treatise on Advanced Calculus, Wiley, Inc. NewY ork.
12. Piskunov, N. (1969) Differential and Integral Calculus, MIR Publishers, Moscow.
13. Prasad, Chandrika (1967). Mathematics for Engineers, Pothishala Private Ltd.
14. Ram, Babu. Engineering Mathematics, Pearson.
15. Thomas, G.B. and Finney, R.L. (1992). Calculus and Analytic Geometry, Addision-Wesley, Reading, MA.
16. Tolstov, G.P. (1976). Dover, New York.

4 Multivariable Calculus I

UNIT SPECIFICS

Multivariable Calculus incorporates the concept of limit, continuity and partial derivatives, directional derivatives, total derivative, tangent plane and normal line, maxima, minima and saddle points, method of Lagrange multipliers, gradient, curl and divergence. In this unit we have discussed these topics. Emphasis is on understanding the applicability with the help of solved and unsolved examples.

RATIONALE

Each of the topics in the given unit plays a crucial role in our day-to-day life. Ranging from measuring a temperature, acceleration while driving, maximising the profit of an industry to the flow of fluid while we are in space or a flight, every single aspect requires application of the mentioned topics in the unit. Not only mathematically, but these topics have a huge significance in the business and economics, industry, biology and medicine, architecture, physical science and engineering, and many other domains. Limits are the only way in mathematics to deal with infinite. Maxima-Minima has a significance in estimating the initial velocity and the launch angle of a projectile (an object which is thrown) to maximize its height/range. The method of Lagrange multipliers is a powerful tool which can be used to solve the class of problems without the need to solve explicitly the conditions and then use them to eliminate extra variables. Also Radio, TV broadcast, Electric motor or dynamo, are designed using Maxwell's equations which is based on gradient, divergence and curl.

PRE-REQUISITES

Students must aware with

1. Functions, limits and continuity in one variable
2. Partially differentiation of a given function
3. Knowledge of various types of Trajectories
4. Sketching of graph for a given curve
5. Concept of maxima-minima in one variable
6. 3-D coordinates, direction of motion in 3-D plane
7. Operations on vectors like addition, subtraction etc.

UNIT OUTCOMES

After completion of this unit, students will be able to-

U4-01: Calculate the limit, examine the continuity and comprehend differentiation geometrically; learn inter-relationship among limit, continuity and differentiability; correlate differentiation with other fields.

U4-02: Learn conceptual advancement from one variable to several variables in calculus; learn the concept of evaluation of derivatives of composite functions with the use of chain rule.

U4-03: Apply multivariable calculus in optimizing problems; formulate and evaluate the complex problems with the help of Lagrange's method of multiplier.

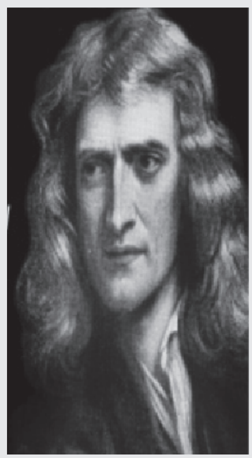
U4-04: Use directional derivatives with gradient, divergence and curl; illustrate the geometric meaning of these with the aid of figures.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 4 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1-Weak Correlation; 2-Medium Correlation; 3-Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U4-01	2	–	–	3	–
U4-02	–	2	2	3	1
U4-03	1	–	–	3	1
U4-04	1	1	1	3	–

HISTORY

Calculus, which generally consists of limits, derivatives, and integrals, is that branch of mathematics that generally deals with the study of change in the value of a function as the points in the domain change. Limit is the value of the function that approaches to some value as the input approaches some value. The current content that comprises calculus has been the result of the efforts of numerous scientists. The founding stones of modern calculus were laid by Gottfried Wilhelm Leibniz (1646–1716) and Sir Isaac Newton (1642–1727). Although developed in the 17th and 18th century, the modern form of calculus dates back to 1817 when Bernard Bolzano gave a formal definition of "epsilon-delta definition of limit" carrying forward the concept of Augustin-Louis Cauchy. Moreover, the definitive modern statement was ultimately provided by Karl Weierstrass. Continuity (which means graphs remains as a single unbroken curve in its domain), Maxima and minima (the largest and the smallest value of the function), divergence (describing the rate of change of a vector in 3-D) are more such parts of Calculus.



What we know is a drop, what we don't is an ocean.

–Isaac Newton

4.1 LIMIT OF A FUNCTION

The idea of the limit of a function of single variable can be extended to the case of the limit of a function of two variables.

Definition. A function $f(x, y)$ is said to have limit 'l' as (x, y) tends to (a, b), if for any arbitrary choosen positive number 'ε', however small, $\exists$ a positive number 'δ' depending upon 'ε', such that for all points (x, y) other than (a, b).

i.e., $|f(x, y) - l| < \varepsilon$ for $0 < |x - a| < \delta$ and $0 < |y - b| < \delta$

and we write it as

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad \text{or} \quad \lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$$

'l' is the limit (the double limit or the simultaneous limit) of $f(x, y)$ when (x, y) tends to (a, b) simultaneously.

4.1.1 Algebra of Limits

Theorem. If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = m$, then

i. $\lim\,[f(x, y) + g(x, y)] = l + m$

ii. $\lim\,[f(x, y) - g(x, y)] = l - m$

iii. $\lim\,[f(x, y) \,.\, g(x, y)] = l \,.\, m$

iv. $\lim\,[k\,f(x, y)] = kl$ [for any no. k]

v. $\lim \dfrac{f(x,y)}{g(x,y)} = \dfrac{l}{m}$, when $(x, y) \to (a, b)$ [provided $m \neq 0$]

The proofs are exactly similar to those of the corresponding theorems for a single variable.

SOME SOLVED EXAMPLES

Example 4.1. *Prove that* $\lim\limits_{(x,y)\to(1,1)} (x^2 + 3y) = 4.$

Solution. Method I: Using definition of limit

We have to show that for any $\varepsilon > 0$, we can find $\delta > 0$, such that

$$|x^2 + 3y - 4| < \varepsilon, \text{ when } |x - 1| < \delta, |y - 1| < \delta$$

If $|x - 1| < \delta$ and $|y - 1| < \delta$, then

$$-\delta + 1 < x < \delta + 1 \quad \text{and} \quad 1 - \delta < y < \delta + 1, \text{ excluding } x = 1, y = 1$$

Thus, $\quad (1 - \delta)^2 < x^2 < (\delta + 1)^2$

or $\quad 1 + \delta^2 - 2\delta < x^2 < 1 + \delta^2 + 2\delta \quad$ ...(A)

and $\quad 3 - 3\delta < 3y < 3\delta + 3 \quad$ ...(B)

Adding (A) and (B), we have

or $\quad \delta^2 - 5\delta + 4 < x^2 + 3y < \delta^2 + 5\delta + 4$

or $\quad \delta^2 - 5\delta < x^2 + 3y - 4 < \delta^2 + 5\delta$

Now, if $\delta \leq 1$, it follows that

$$-6\delta < x^2 + 3y - 4 < 6\delta$$

i.e., $|x^2 + 3y - 4| < 6\delta = \varepsilon$, where $\delta = \dfrac{\varepsilon}{6}$

$\therefore$ $|x^2 + 3y - 4| < \varepsilon$ when $|x - 1| < \delta, |y - 1| < \delta$

$$\therefore \lim_{(x,y)\to(1,1)} (x^2 + 3y) = 4$$

Method II: Using a theorem based on algebra of limits

$$\lim_{(x,y)\to(1,1)} x^2 + 3y = \lim_{(x,y)\to(1,1)} x^2 + \lim_{(x,y)\to(1,1)} 3y$$
$$= 1 + 3 = 4.$$

Example 4.2. *Show that* $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2 + y^6}$ *does not exist.*

Solution. Let $(x, y) \to (0, 0)$ along the path $y^3 = mx$, where 'm' is any real number. As $x \to 0$, then $y \to 0$ along the path $y^3 = mx$.

$$\text{Now,} \quad \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6} = \lim_{x\to 0} \frac{x(mx)}{x^2 + (mx)^2}$$
$$= \lim_{x\to 0} \frac{m}{1 + m^2}$$

which is different for different values of m.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6} \text{ does not exist.}$$

> **Remark:** In single variable if limit from right and left exist and equal, then we write limit exist. Similarly, here if limit exist for every possible path and equal, then we say that limit exists. If there exists two different paths for which limit not equal, then we say limit does not exists.

Example 4.3. *Show that* $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$, *does not exist.*

Solution. If we take path $y = x$, then $\lim\limits_{x\to 0} \dfrac{x \,.\, x}{x^2 + x^2} = \dfrac{1}{2}$

If we take path $y = 2x$, then $\lim\limits_{x\to 0} \dfrac{x \,.\, 2x}{x^2 + 4x^2} = \dfrac{2}{5}$

Here, for two different paths, limit is not same.

Hence, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

Example 4.4. *Show that* $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0,$

where $f(x, y) = x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}, x \neq 0, y \neq 0.$

Solution. Let $\varepsilon > 0$ be any given real number.

Now, $$|f(x,y)-0| = \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \le \left|x\sin\frac{1}{y}\right| + \left|y\sin\frac{1}{x}\right|$$

$$\le |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right|$$

$$\le |x| + |y| \qquad \left[\because \left|\sin\frac{1}{x}\right| \le 1 \text{ and } \left|\sin\frac{1}{y}\right| \le 1\right]$$

$$= |x-0| + |y-0| \qquad ...(1)$$

Let $|x-0| < \frac{\varepsilon}{2}$ and $|y-0| < \frac{\varepsilon}{2}$

From (1), $$|f(x,y)-0| \le |x-0| + |y-0|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $$|f(x,y)-0| < \varepsilon \quad \text{when } |x-0| < \frac{\varepsilon}{2} \text{ and } |y-0| < \frac{\varepsilon}{2}$$

Taking $\delta = \frac{\varepsilon}{2}$, we have

$$|f(x,y)-0| < \varepsilon \quad \text{when } |x-0| < \delta \text{ and } |y-0| < \delta$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Example 4.5. *Show that* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

Solution. Let $\varepsilon > 0$ be any given real number.

Now, $$|f(x,y)-0| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| \Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}}\right|$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we get, $x^2 + y^2 = r^2$

$$|f(x,y)-0| = \left|\frac{r^2\cos\theta\sin\theta}{\sqrt{r^2\cos^2 + r^2\sin^2\theta}}\right|$$

$$= \left|\frac{r^2(\cos\theta\,.\,\sin\theta)}{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}}\right|$$

$$= \left|\frac{r^2}{r}(\cos\theta\,.\,\sin\theta)\right|$$

$$= |r\cos\theta\,.\,\sin\theta|$$

$$\le |r| = r \qquad [\because |\cos\theta| \le 1, |\sin\theta| \le 1]$$

$$= \sqrt{x^2+y^2} \qquad [\because r^2 = x^2 + y^2]$$

$\therefore$ $$|f(x,y)-0| < \varepsilon \quad \text{if} \quad \sqrt{x^2+y^2} < \varepsilon$$

i.e., if $x^2 + y^2 < \varepsilon^2$

i.e., if $|x^2| < \dfrac{\varepsilon^2}{2}, |y^2| < \dfrac{\varepsilon^2}{2}$

i.e., if $|x| < \dfrac{\varepsilon}{\sqrt{2}}, |y| < \dfrac{\varepsilon}{\sqrt{2}}$

Taking $\dfrac{\varepsilon}{\sqrt{2}} = \delta$, we have $|f(x, y) - 0| < \varepsilon$, when $|x - 0| < \delta$ and $|y - 0| < \delta$

$$\Rightarrow \quad \lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.$$

4.1.2 Repeated Limit

If a function $f(x, y)$ is defined in some neighbourhood of (a, b), then the limit $\lim\limits_{x \to a} f(x, y)$, if it exists, is a function of y, say, $\phi(y)$. Then, if the limit $\lim\limits_{y \to b} \phi(y)$ exists, and is equal to λ, we write,

$\lim\limits_{y \to b} \lim\limits_{x \to a} f(x, y) = \lambda$ and say that λ is a repeated limit of $f(x, y)$ as $x \to a, y \to b$.

If we change the order of applying the limits, we get the other repeated limit,

$\lim\limits_{x \to a} \lim\limits_{y \to b} f(x, y) = \lambda_1$ (say) where first $y \to b$ and then $x \to a$.

These two limits may or may not equal.

SOME SOLVED EXAMPLES

Example 4.6. *Find the repeated limit and double limit of* $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$ *at (0, 0).*

Solution. Let $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$, then

$\lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x, y) = \lim\limits_{y \to 0} (0) = 0$, Similarly, $\lim\limits_{x \to 0} . \lim\limits_{y \to 0} f(x, y) = \lim\limits_{x \to 0} (0) = 0$

Thus the repeated limits exists and are equal. But the double limit does not exists when putting $y^3 = mx$ as shown in example 4.2.

> **Remark:** If the repeated limits are not equal, then simultaneous (double) limit cannot exists but converse is not true.

Example 4.7. *Find repeated limits and double limit of the function* $f(x, y) = \dfrac{(y - x)}{(y + x)} \cdot \dfrac{(1 + x)}{(1 + y)}$ *at (0, 0).*

Solution. Let $f(x, y) = \dfrac{(y - x)}{(y + x)} \cdot \dfrac{(1 + x)}{(1 + y)}$

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \frac{(-x)}{(x)} (1 + x) = -1$$

and $$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}\frac{y}{y}\frac{(1)}{(1+y)} = 1$$

Double limit

$$\lim_{(x, y)\to(0,0)} f(x, y) = \lim_{(x, y)\to(0,0)}\frac{(y-x)}{(y+x)}\cdot\frac{(1+x)}{(1+y)} \qquad ...(2)$$

Putting $y = mx$, then from (2)

$$\lim_{(x, y)\to(0,0)} f(x, y) = \lim_{x\to 0}\frac{(mx-x)}{(mx+x)}\cdot\frac{(1+x)}{(1+xm)}$$

$$= \frac{m-1}{m+1}$$

which is different for different values of m.

Thus, the two repeated limits exists but are not equal, consequently, the simultaneous limit cannot exist.

4.1.3 Continuity of a Function

A function $f(x, y)$ is said to be continuous at a point (a, b) if for any given $\varepsilon > 0$, we can find a real number $\delta > 0$ (depending upon ε) such that $|f(x, y) - f(a, b)| < \varepsilon$ for all x, y that satisfy $|x - a| < \delta$ and $|y - b| < \delta$.

4.1.4 Algebra of Continuous Function

Theorem: If $f(x, y)$ and $g(x, y)$ are continuous function of x and y on a set D and k is any constant, then

i. $f(x, y) + g(x, y)$ is continuous on D.

ii. $f(x, y) - g(x, y)$ is continuous on D.

iii. $f(x, y) \,.\, g(x, y)$ is continuous on D.

iv. $kf(x, y)$ is continuous on D.

v. $\dfrac{f(x, y)}{g(x, y)}$ is continuous on D except for those points of (x_0, y_0) where $g(x_0, y_0) = 0$

The proofs of the above are exactly similar to those of the corresponding theorem for a single variable.

SOME SOLVED EXAMPLES

Example 4.8. *Show that the function*

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x, y)\neq(0,0) \\ 0, & (x, y)=(0,0) \end{cases}$$

is continuous at origin.

Solution. Let $f(x, y) = \dfrac{x^2y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$

Putting $x = r\cos\theta$, $y = r\sin\theta$ we get $x^2 + y^2 = r^2$

$$\therefore \qquad \left|\frac{x^2y^2}{x^2+y^2}-0\right| = \left|\frac{r^4\cos^2\theta\sin^2\theta}{r^2}\right|$$
$$= |\,r^2\cos^2\theta.\sin^2\theta\,|$$
$$\le r^2 \qquad [\because \quad |\sin\theta| \le 1 \text{ and } |\cos\theta| \le 1]$$
$$= x^2+y^2$$

$$\therefore \qquad \left|\frac{x^2y^2}{x^2+y^2}-0\right| < \varepsilon \text{ if } x^2+y^2 < \varepsilon$$

if $\qquad |x^2| < \dfrac{\varepsilon}{2}, |y^2| < \dfrac{\varepsilon}{2}$

i.e., if $\qquad |x| < \sqrt{\dfrac{\varepsilon}{2}}, |y| < \sqrt{\dfrac{\varepsilon}{2}}$

Taking $\qquad \sqrt{\dfrac{\varepsilon}{2}} = \delta$, we have

$$\lim_{(x,y)}.\frac{x^2y^2}{x^2+y^2} = 0 = f(0,0)$$

Hence, $f(x, y)$ is continuous at (0, 0).

Example 4.9. *Investigate the continuity at (0, 0), for the given function,*

$$f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^4+y^4}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Solution. Try yourself.

Example 4.10. *Investigate the continuity at (2, 3) for the given function*

$$f(x,y) = \begin{cases} 3xy, & (x,y) \ne (2,3) \\ 6, & (x,y) = (2,3) \end{cases}.$$

Solution. Here, $\qquad f(x, y) = 3xy, (x, y) \ne (2, 3)$

$$\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} 3xy = 18 \ne f(2,3)$$

Hence, $f(x, y)$ is not continuous at (2, 3).

Note: If $f(2, 3) = 18$, then $f(x, y)$ is continuous at (2, 3). Hence $f(x)$ has removable discontinuity at (2, 3).

Example 4.11. *Prove that the function*

$$f(x,y) = \begin{cases} y\sin\dfrac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at (0, 0).

Solution. Let $\varepsilon > 0$ be any given number, then

$$|f(x,y) - f(0,0)| = \left|y\sin\frac{1}{x} - 0\right|$$

$$= \left| y \sin\frac{1}{x} \right| = |y| \left| \sin\frac{1}{x} \right|$$

$$\leq |y| \qquad \left[\because \left|\sin\frac{1}{x}\right| \leq 1\right]$$

$\therefore \qquad |f(x, y) - f(0, 0)| \leq |y - 0| \qquad ...(1)$

Let $\qquad |x - 0| < \varepsilon$ and $|y - 0| < \varepsilon$

From (1), we have $|f(x, y) - f(0, 0)| < \varepsilon$

Hence, $f(x, y)$ is continuous at $(0, 0)$.

EXERCISE 4.1

1. Show that $\lim\limits_{(x, y)\to(0,0)} (x + y) = 0$.

2. Evaluate the limit of the following:

i. $\lim\limits_{(x, y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2}$

ii. $\lim\limits_{(x, y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2 + (x^2 - y^2)^2}$

iii. $\lim\limits_{(x, y)\to(0,0)} \dfrac{x^3 + y^3}{x - y}$ (**Hint:** Put $y = x - mx^3$)

3. Show that

i. $\lim\limits_{(x, y)\to(0,0)} \dfrac{x\sin(x^2 + y^2)}{x^2 + y^2} = 0$

ii. $\lim\limits_{(x, y)\to(0,0)} \dfrac{\sin^{-1}(xy - 2)}{\tan^{-1}(3xy - 6)} = \dfrac{1}{3}$

iii. $\lim\limits_{(x, y)\to(0,1)} \dfrac{x + y - 1}{\sqrt{x} - \sqrt{1 - y}} = 0$

iv. $\lim\limits_{(x, y)\to(3,2)} (x + y)\dfrac{\sin^{-1}(xy - 6)}{\cos^{-1}(2xy - 12)} = 0$

4. Discuss the continuity of the following functions at origin

i. $f(x, y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

ii. $f(x, y) = \begin{cases} \dfrac{x^3y^3}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

iii. $f(x, y) = \begin{cases} 2xy \cdot \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$ [**Hint:** $y = x - mx$]

iv. $f(x, y) = \begin{cases} \dfrac{xy^3}{x^2 + y^6}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

5. Show that the function $f(x, y) = \begin{cases} 1, & x \text{ is irrational} \\ 0, & x \text{ is rational} \end{cases}$ is continuous nowhere.

6. Show that the function $f(x, y) = \begin{cases} 1, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$ is not continuous at (a, b) if $b = 0$.

7. Show that the function $f(x, y) = \begin{cases} x \sin \dfrac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$ is continuous at $(0, 0)$.

8. If $f(x, y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^{5/2}}[1 - \cos(x^2 + y^2)], & (x, y) \neq (0, 0) \\ k, & (x, y) = (0, 0) \end{cases}$ then, find the value of k for which $f(x, y)$ is continuos at $(0, 0)$.

9. If $f(x, y) = \begin{cases} \dfrac{\sin\{2(x^2 + y^2)\, e^{3x \sin(4/y)}}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ \alpha, & (x, y) = (0, 0) \end{cases}$. Find the values of 'α' for which $f(x, y)$ is continuous at $(0, 0)$.

Continuity of Function of Two Variable

Answers

2. i. 0 ii. does not exist iii. does not exist

4. i. not continuous at $(0, 0)$ ii. continuous at $(0, 0)$

iii. continuos at $(0, 0)$ iv. not continuous at $(0, 0)$

INTERESTING FACTS (LIMIT+CONTINUITY)

- As driver drive a car, its velocity continues to increase, which is termed as "Acceleration". It is a concept that is used to describe how velocity changes over time. Velocity and acceleration are measured using a fundamental concept of calculus which is called the derivative.
- Tangents can find using derivatives that have an important use in the domain of telescope manufacturing (where angle needs to be taken care of).
- Time is considered to be continuous (but the introduction of Quantum Physics, time and space have been termed as non-continuous).

REAL LIFE EXAMPLES (LIMIT AND CONTINUITY)

- In chemistry, finding the pressure at a point with the help of concept of limit (since its force per unit area within the diameter is shrinking to zero).
- Helps in calculating the velocity of falling object at any instant of time.
- In Physics, the concept of Thermodynamics uses the concept of limits, since for macroscopic equilibrium to reach, one must stop for indefinite period of time.
- Acceleration is an example of continuity. It is not like, it drops from 50 to 2 immediately (there are continuous values in between).

4.2 DIRECTIONAL DERIVATIVE

Here, we introduce a type of derivative, called a directional derivative, that enables us to find the rate of change of a function of two or more variables in any direction.

Definition: Let $S \subseteq R^m$ and let $f : S \to R^m$ be a function defined on S with values in R^m. Then the directional derivative of 'f' at 'c' in the direction of 'u' is denoted as $f'(c, u)$ and defined as

$$D_u f(c) = f'(c, u) = \lim_{h \to 0} \frac{f(c+hu) - f(c)}{h}$$

SOME SOLVED EXAMPLES

Example 4.12. *Find the directional derivative along* $(\sqrt{2}, \sqrt{2})$ *at (0, 0) for*

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Solution. Here,
$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Now,
$$\begin{aligned} f'(c, u) &= \lim_{h \to 0} \frac{f(c+hu) - f(c)}{h} \\ &= \lim_{h \to 0} \frac{f((0,0) + h(\sqrt{2}, \sqrt{2})) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{f(h\sqrt{2}, h\sqrt{2}) - 0}{h} \\ &= \lim_{h \to 0} \frac{1}{h}\left[\frac{h\sqrt{2}\,.(h\sqrt{2})^2}{2h^2 + 4h^4}\right] \\ &= \lim_{h \to 0} \frac{1}{h} \frac{h^3\,.2\sqrt{2}}{2h^2 + 4h^4} \\ &= \lim_{h \to 0} \frac{h^3}{h^3} \cdot \frac{2\sqrt{2}}{2(1+2h^2)} \\ &= \lim_{h \to 0} \frac{\sqrt{2}}{(1+2h^2)} = \frac{\sqrt{2}}{1} = \sqrt{2} \end{aligned}$$

Example 4.13. *Find the directional derivative of f(x, y) along (a, b), a ≠ 0, b ≠ 0 at (0, 0) for*

$$f(x, y) = \begin{cases} x + y, & x \text{ or } y \neq 0 \\ 0 & \text{else} \end{cases}$$

Solution. Here,
$$f(x, y) = \begin{cases} x + y, & x \text{ or } y \neq 0 \\ 0 & \text{else} \end{cases}$$

then, $$f'(c,u) = \lim_{h\to 0} \frac{f((0,0)+h(a,b))-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{ha+hb}{h} = a+b$$

Example 4.14. *Find the directional derivative of f(x, y) at (0, 0) along the direction* $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ *for*

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Solution. For the given $f(x, y)$, we have the directional derivative as

$$f'(c,u) = \lim_{h\to 0} \frac{f\left((0,0)+h\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right)-0}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} f\left(\frac{h}{\sqrt{2}},\frac{h}{\sqrt{2}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\frac{h^2}{2}\cdot\frac{h}{\sqrt{2}}}{\frac{h^4}{4}+\frac{h^2}{2}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\frac{h^3}{2\sqrt{2}}}{\frac{h^4+2h^2}{4}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h^3}{2\sqrt{2}}\times\frac{4}{h^2\,(2+h^2)}\right] = \lim_{h\to 0}\left[\frac{2}{\sqrt{2}(2+h^2)}\right]$$

$$= \frac{1}{\sqrt{2}}$$

Remark 1: If $f:\mathbb{R}^n \to \mathbb{R}^m$, such that $f(x_1, x_2, ..., x_n) = (f_1, f_2, f_3, ..., f_m)$ then $f'(c, u)$ exist iff $f_i\,(c, u)$ exists $\forall\ i = 1$ to m.

For example: If $f: R^2 \to R^2$ be such that $f(x, y) = (3x - 2y + x^2, 4x + 3y + y^2)$, find directional derivative at (0, 0) in the direction (a, b).

Solution: Here, $$f'(c,u) = \lim_{h\to 0} \frac{f(c+hu)-f(c)}{h}$$

$$= \lim_{h\to 0} \frac{f(ha,hb)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{(3ha-2hb+h^2a^2,\, 4ha+3hb+h^2b^2)}{h}$$

$$f'(c, u) = \lim_{h \to 0} (3a - 2b + ha^2, 4a + 3b + hb^2)$$
$$= (3a - 2b, 4a + 3b)$$

Hence directional derivative exist in every direction
(As limit exist in every direction)

Remark 2: The existence of directional derivative does not ensure the continuity.

For example: If $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Solution. For the given function, directional derivative exists along (1, 0) at (0, 0).

Let $\quad c = (0, 0), u = (1, 0)$

then $\quad f'(c, u) = \lim_{h \to 0} \dfrac{f[(0,0) + h(1,0)] - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{f(h,0) - 0}{h} = 0$$

and $\quad \lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4} = \lim_{y \to 0} \dfrac{m}{1 + m^2}$ [by putting $x = my^2$]

Hence, $\quad \lim_{(x,y)\to(0,0)} f(x, y) = \dfrac{m}{1 + m^2}$ does not exist [Path dependent]

Here, directional derivative exist, but f is not continuous, even limit does not exist at (0, 0).

Result: A function $f: \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x) = (f_1, f_2, ..., f_m)$ and if $f(x)$ satisfies the condition

$$f(\alpha x + \beta y) = \alpha . f(x) + \beta f(y) \qquad \text{(linear function } \forall\, x, y \in \mathbb{R}^n \text{ and } \forall\, \alpha, \beta \text{ scalars}$$

then directional derivatives of 'f' at 'c' in the direction of 'u' is given by

$$f'(c, u) = f(u)$$

i.e., if 'f' is linear map, then

$$\lim_{h \to 0} \frac{f(c + hu) - f(c)}{h} = \lim_{h \to 0} \frac{f(c) + hf(u) - f(c)}{h}$$
$$= \lim_{h \to 0} f(u)$$

$\Rightarrow \quad f'(c, u) = f(u)$

For example: Let $f: R^2 \to \mathbb{R}^2$ be such that $f(x, y) = (x, -y)$. Find directional derivative at (a, b) in the direction of (1, 0).

Solution: Since 'f' satisfies linear property.

$\therefore \quad f'(c, u) = f(u)\ \forall\, c \in R^n$

or $\quad f'(c, u) = f(1, 0) = (1, 0)$

4.2.1 Directional Derivatives in Terms of Partial Derivatives

Let $f: R^2 \to R$ and $f(x, y)$ be a given function, then $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ means that the directional derivative in the direction of (1, 0) and (0, 1) *i.e.*, in the direction of x-axis and y-axis respectively.

Directional derivative at $c = (a, b)$, in the direction of $(1, 0)$

i.e., $c = (a, b) \in R^2$ and $u = (1, 0)$

$$f'(c, u) = \lim_{h \to 0} \frac{f(c + hu) - f(c)}{h}$$

$$= \lim_{h \to 0} \frac{f(c + h(1, 0)) - f(c)}{h}$$

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = f'(c, u) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the directional derivative at $c = (a, b)$ in the direction of $(0, 1)$ *i.e.,* $u = (0, 1)$ and $c = (a, b) \in \mathbb{R}^2$

then
$$\left(\frac{\partial f}{\partial y}\right)_{(a,b)} = f'(c, u) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

In general, if $f : \mathbb{R}^n \to R$ is one valued function, then the partial derivative $\dfrac{\partial f}{\partial x_k}$ at the point 'c' is the directional derivative in the direction of u_k where u_k is unit vector in the direction of k^{th} axis.

i.e.,
$$u_k = [0, 0, \ldots, \underset{\uparrow\, k\text{th position}}{1}, 0, 0, \ldots]$$

i.e., $f'(c, u_k)$ is the partial derivative of 'f' in the direction of u_k.

i.e., $f'(c, u_k) = \dfrac{\partial f}{\partial x_k}$ at c

> **Remark:** Existence of directional derivatives in all directions clearly implies the existence of partial derivatives. However, converse is not true *i.e.*, existence of partial derivatives does not imply the existence of directional derivatives in all directions.

For example: Show that for the function $f : R^2 \to R$ defined by $f(x, y) = \begin{cases} x + y, & \text{if } x = 0 \text{ or } y = 0 \\ 1, & \text{otherwise} \end{cases}$

both first order partial derivatives exists at $(0, 0)$ but directional derivatives does not exist in all directions at $(0, 0)$.

Solution. Here $D_1 f(0, 0) = f_x(0, 0)$

$$= \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

and $D_2 f(0, 0) = f_y(0, 0)$

$$= \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

But, if consider any other direction $u = (u_1, u_2)$, where $u_1 \neq 0$ and $u_2 \neq 0$, then

$$D_y f(0, 0) = \lim_{h \to 0} \frac{f(hu_1, hu_2) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}, \text{ which does not exist.}$$

EXERCISE 4.2

1. Let $f : R^2 \to R$ be a function defined by $f(x, y) = 3x + 4y + x^2$. Then find the directional derivative of 'f', for the given:

i. at (0, 0) in the direction (1, 2)
ii. at (1, –1) in the direction (1, –1)
iii. at (1, 2) in the direction (1, –3)
iv. at (1, –1) in the direction (1, 0)
v. at (1, –1) in the direction (0, 1)
vi. at (c_1, c_2) in the direction (u_1, u_2)

2. Let $f : R^2 \to R^2$ such that $f(x, y) = (x, -y)$. Find the directional derivative of 'f' at (a, b) in the direction of (2, 3).

3. Let $f(x, y) = \begin{cases} x + y, & \text{if } x = 0 \text{ or } y = 0 \\ 1, & \text{otherwise} \end{cases}$

Find the directional derivative of 'f' at (0, 0) in the direction of $(a, 0)$, $a \neq 0$.

4. Let $f : R^2 \to R^2$ defined by $f(x, y) = (7x + x^4, 3x + 4y + y^4)$, then find the directional derivative of 'f' at (0, 0) in the direction of (a, b).

Answers

1. i. 11 ii. 1 iii. –7 iv. 5
v. 4 vi. $3u_1 + 4u_2 + 2c_1u_1$

2. (2, –3) **3.** a **4.** $7a, 3a + 4b$

4.3 PARTIAL DERIVATIVES

If z be a function of two independent variables x, y *i.e.* if $z = f(x, y)$, then the derivative of z w.r.t. x, treating y as constant, is called the partial derivative of z w.r.t. x and is denoted by the symbol $\frac{\partial z}{\partial x}$. Similarly when x is treated as constant, we may find the partial derivative of z with respect to y, which we shall denote by the symbol $\frac{\partial z}{\partial y}$. Thus

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x, y)] = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x, y)] = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

provided these limit exists.

The following symbols are commonly used to denote the partial derivatives of a function of two variables.

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, \frac{\partial}{\partial x}[f(x, y)], f_x(x, y)$$

and
$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, \frac{\partial}{\partial y}[f(x, y)], f_y(x, y)$$

In general, if 'u' is a function of n independent variables x_1, x_2, x_3, ... x_n, then the partial derivatives of u with respect to any variable x_r, $r = 1, 2, ... n$, is obtained by differentiating u, w.r.t. x_r, treating all other variables independent of x_r *i.e.*, as constant.

4.3.1 Partial Derivatives of Higher Orders

If $z = f(x, y)$, then the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are usually functions of both x and y (though one or both of them may be either constant or dependent on only one of the variables) they may possess partial derivatives with respect to x and y. These partial derivatives are called the second order partial derivatives of z.

These derivatives are commonly denoted by one of the following symbols:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy};$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx};$$

Partial derivatives of higher order are defined in the similar manner.

> **Remark:** It should be noted that $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are, in general, not equal. If however, the function possesses continuous partial derivatives of second order, then, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and this property is known as the commutative property of the partial derivatives.

SOME SOLVED EXAMPLES

Example 4.15. *Find* $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *when* $u = \log_e (x^2 + y^2)$.

Solution. Here $u = \log_e (x^2 + y^2)$ ...(i)

Differentiating (i) partially w.r.t. x (treating y as constant), we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} 2x = \frac{2x}{x^2+y^2}$$

Differentiating (i) partially w.r.t. y (treating x as constant), we get

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} . 2y = \frac{2y}{x^2+y^2}$$

Example 4.16. *Verify that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, *where* $u = \log_e \left(\frac{x^2+y^2}{xy}\right)$.

Solution. Here $u = \log_e \left(\frac{x^2+y^2}{xy}\right)$

or $u = \log_e (x^2 + y^2) - \log_e x - \log_e y$ ... (1)

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} . 2x - \frac{1}{x} = \frac{2x}{x^2+y^2} - \frac{1}{x} \qquad ...(2)$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} . 2y - \frac{1}{y} = \frac{2y}{x^2+y^2} - \frac{1}{y} \qquad ...(3)$$

Differentiating (2) partially w.r.t. y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - \frac{1}{x}\right) = 2x \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$= \frac{-2x}{(x^2+y^2)^2} . 2y$$

i.e., $$\frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2+y^2)^2} \qquad ...(4)$$

Differentiating (3) partially w.r.t. x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2} - \frac{1}{y}\right) = 2y \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= \frac{-2y}{(x^2+y^2)^2} . 2x$$

i.e., $$\frac{\partial^2 u}{\partial x \partial y} = \frac{-4xy}{(x^2+y^2)^2} \qquad ...(5)$$

$\therefore$ From (4) and (5), we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$ **Proved**

Example 4.17. *If* $u (x + y) = (x^2 + y^2)$, *then show that* $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Solution. Try yourself.

Example 4.18. *If* $u = f(y + ax) + \phi(y - ax)$, *prove that* $\dfrac{\partial^2 u}{\partial x^2} - a^2 \dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution. Try yourself.

Example 4.19. *If* $u = \log_e (x^3 + y^3 + z^3 - 3xyz)$, *then show that* $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$.

Solution. Since $u = \log_e (x^3 + y^3 + z^3 - 3xyz)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{(3x^2 - 3yz)}{(x^3 + y^3 + z^3 - 3xyz)} \qquad ...(1)$$

and
$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{(x^3 + y^3 + z^3 - 3xyz)} \qquad ...(2)$$

and
$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{(x^3 + y^3 + z^3 - 3xyz)} \qquad ...(3)$$

Adding (1), (2) and (3), we get

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)}$$

$$= \frac{3}{(x+y+z)} \qquad ...(4)$$

Now
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \qquad \text{[by (4)]}$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$ **Proved.**

Example 4.20. If $u = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\dfrac{y}{x}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Solution. Try yourself.

Example 4.21. *If* $x^x y^y z^z = \lambda$, *show that at* $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log_e xe)^{-1}.$$

Solution. Given that $x^x y^y z^z = \lambda$, where z is a function of x and y.

Taking logarithms on both sides, we get

$$x \log x + y \log y + z \log z = \log \lambda \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$\left[\log x + x \cdot \frac{1}{x}\right] + \left[\log z + z \cdot \frac{1}{z}\right] \cdot \frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)} \qquad ...(2)$$

Differentiating (1) partially w.r.t. y, we get

$$\left[\log y + y \cdot \frac{1}{y}\right] + \left[\log z + z \cdot \frac{1}{z}\right] \cdot \frac{\partial z}{\partial y} = 0$$

or $$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)} \qquad ...(3)$$

Now, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[-\frac{(1+\log x)}{(1+\log z)}\right]$$

$$= -(1+\log x)\frac{\partial}{\partial y}\left(\frac{1}{(1+\log z)}\right)$$

$$= -(1+\log x)\left[-\frac{1}{(1+\log z)^2}\frac{1}{z}\frac{\partial z}{\partial y}\right]$$

$$= \frac{1}{z}\frac{(1+\log x)}{(1+\log z)^2}\left[-\frac{(1+\log y)}{(1+\log z)}\right] \qquad \text{[from 3]}$$

$$= \frac{-(1+\log x)(1+\log y)}{z \cdot (1+\log z)^3}$$

$\therefore$ At $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x[\log_e e + \log_e x]} = -\frac{1}{x \log_e x e}$$

Thus $$\frac{\partial^2 z}{\partial x \partial y} \text{ (at } x = y = z) = -[x \log_e x e]^{-1}$$ **Proved.**

Example 4.22. *If* $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$, *then show that*

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4y^2 \sin^{-1}\left(\frac{x}{y}\right).$$

Solution. Since $$z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = x^4y^2 \cdot \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y} + 4x^3y^2 \sin^{-1}\left(\frac{x}{y}\right) + \frac{1}{x}$$

$$x\frac{\partial z}{\partial x} = x^5y^2 \cdot \frac{1}{\sqrt{y^2-x^2}} + 4x^4y^2 \sin^{-1}\left(\frac{x}{y}\right) + 1 \qquad ...(2)$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = x^4y^2.\ \frac{1}{\sqrt{1-\dfrac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + 2x^4y\sin^{-1}\left(\frac{x}{y}\right) - \frac{1}{y}$$

or $$y\frac{\partial z}{\partial y} = -x^5y^2.\frac{1}{\sqrt{y^2-x^2}} + 2x^4y^2\sin^{-1}\left(\frac{x}{y}\right) - 1 \qquad ...(3)$$

Adding (2) and (3), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4y^2\sin^{-1}\left(\frac{x}{y}\right) \quad \textbf{Proved.}$$

Example 4.23. *Find* $\frac{\partial z}{\partial r}$ *and* $\frac{\partial z}{\partial \theta}$, *if* $z = e^{r\cos\theta} . \cos(r\sin\theta)$.

Solution. Given that $\quad z = e^{r\cos\theta} . \cos(r\sin\theta) \qquad ...(1)$

Differentiating (1) partially w.r.t. r, we get

$$\frac{\partial z}{\partial r} = e^{r\cos\theta}\left[\{-\sin(r\sin\theta)\sin\theta\} + \{\cos(r\sin\theta) . \cos\theta\}\right]$$

$[\because \quad \cos(x+y) = \cos x\cos y - \sin x\sin y]$

$$= e^{r\cos\theta} . \cos(r\sin\theta + \theta) \qquad ...(2)$$

Differentiating (1) partially w.r.t. θ, we get

$$\frac{\partial z}{\partial \theta} = e^{r\cos\theta}[-\sin(r\sin\theta)\, r\cos\theta] + [-r\sin\theta\, e^{r\cos\theta}] . \cos(r\sin\theta)$$

$$= -r\,e^{r\cos\theta}[\sin(r\sin\theta)\cos\theta + \sin\theta\cos(r\sin\theta)]$$

$$= -r\,e^{r\cos\theta}[\sin(r\sin\theta + \theta)] \quad \textbf{Answer.}$$

Example 4.24. *If* $u = f(r)$, *where* $r^2 = x^2 + y^2$, *show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

Solution. Given that, $\quad r^2 = x^2 + y^2 \qquad ...(1)$

Differentiating (1) partially w.r.t. x, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \qquad ...(2)$$

Similarly $$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad ...(3)$$

Now $\quad u = f(r)$, given

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} = f'(r) . \frac{x}{r} \qquad \text{[from (2)]}$$

Differentiating both sides partially w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'(r).\frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r\left[f'(r).1 + x.f''(r).\frac{\partial r}{\partial x}\right] - xf'(r).\frac{\partial r}{\partial x}}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[r\,f'(r) + x^2\,f''(r) - \frac{x^2}{r}\,f'(r)\right] \qquad \text{[from (2)]}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[r\,f'(r) + y^2\,f''(r) - \frac{y^2}{r}\,f'(r)\right]$$

Adding (4) and (5), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2r\,f'(r) + (x^2 + y^2)\,f''(r) - \frac{(x^2 + y^2)}{r}\,f'(r)\right]$$

$$= \frac{1}{r^2}\left[2r\,f'(r) + r^2 f''(r) - \frac{r^2}{r}\,f'(r)\right] \qquad \text{[from (1)]}$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}\,f'(r)$$ **Proved.**

Example 4.25. *If* $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$. *Prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right].$$

Solution. Try yourself.

Example 4.26. *If* $\theta = t^n e^{-r^2/4t}$ *for what value of 'n', we have* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$.

Solution. Try yourself, **Answer:** $= \frac{-3}{2}$

Example 4.27. *If* $u = x^y$, *then show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial x}$.

Solution. Try yourself.

Example 4.28. *If* $u = e^{xyz}$, *then show that* $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

Solution. Let $u = e^{xyz}$

$\therefore$ $$\frac{\partial u}{\partial z} = xye^{xyz} \qquad ...(1)$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = x\frac{\partial}{\partial y}\,(y\,e^{xyz})$$

$$= x\,[y\,.\,z\,.\,x\,.\,e^{xyz} + e^{xyz}]$$

$$= e^{xyz}\,[x^2yz + x] \qquad ...(2)$$

Differentiating (2) partially w.r.t. x, we get

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}[e^{xyz}(x^2yz + x)]$$
$$= e^{xyz}(2xyz + 1) + yz\, e^{xyz}(x^2yz + x)$$
$$= e^{xyz}(1 + 2xyz + x^2y^2z^2 + xyz)$$
$$= e^{xyz}(1 + 3xyz + x^2y^2z^2) \quad \textbf{Proved.}$$

EXERCISE 4.3

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, when

i. $u = x \sin y + y \sin x$ ii. $u = \frac{1}{\sqrt{x^2 + y^2}}$

iii. $u = x^y$ iv. $u = \sin^{-1} x/y$

2. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

3. If $u = f\left(\frac{y}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

4. If $u = \sin^{-1}\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, then show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$.

5. If $u = e^x(x \cos y - y \sin y)$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

6. If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$, then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2 + y^2)^{3/2}}$.

7. If $u = (1 - 2xy + y^2)^{-1/2}$, then prove that $\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0$.

8. Find p and q, if $\begin{cases} x = \sqrt{a}\,(\sin u + \cos v) \\ y = \sqrt{a}\,(\cos u - \sin v) \\ z = 1 + \sin(u + v) \end{cases}$

where p and q are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively. **Hint:** $z = \frac{1}{2a}(x^2 + y^2)$

9. If $f(u) = \sin u$ and $u = \sqrt{x^2 + y^2}$, then show that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \cos^2 u$.

10. Prove that, $f(x, y) = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$, then $f_{xy}(x, y) = f_{yx}(x, y)$.

11. If $v = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$ then show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

12. If $v = e^{xyz} + \left(\frac{xz}{y}\right)$, then prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2xyz, y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 2xyzv$

Also deduce that $x\frac{\partial^2 v}{\partial z \partial x} \neq y\frac{\partial^2 v}{\partial z \partial y}$.

13. If $u = x\,\phi\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$, then show that

i. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\,\phi\left(\frac{y}{x}\right)$

ii. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

14. If $v = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, then prove that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$, where v is a function of x and y.

Answers

1. i. $\sin y + y\cos x;\ x\cos y + \sin x$

ii. $\frac{-x}{(x^2+y^2)^{3/2}};\frac{-y}{(x^2+y^2)^{3/2}}$

iii. $x^y.\ \frac{y}{x};\ x^y\log x$

iv. $\frac{-1}{\sqrt{y^2-x^2}};-\frac{x}{y\sqrt{y^2-x^2}}$

8. $p = x/a$ and $q = y/a$.

4.4 HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be a homogeneous function of degree (or order) n, if each term in x and y is of same degree n.

Thus the expression, $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n$, is a homogeneous function of degree n. This expression can also be written as $x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ... + a_n\left(\frac{y}{x}\right)^n\right]$ which can be denoted by the symbol $x^nf\left(\frac{y}{x}\right)$. Hence to enlarge the concept of homogeneity so as to bring even transcendental functions within its scope, we define a homogeneous function as follows.

A function of two variables x, y is said to be a homogeneous function of degree n in x and y if it can be expressed in the form $x^nf(y/x)$.

Thus, for example, the expression, $ax^2 + by^2 + 2hxy$, *i.e.* $x^2\left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right]$ is the homogeneous function of degree 2.

4.4.1 Euler's Theorem

Statement: If u be a homogeneous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Proof: Since u is a homogeneous function of degree n in x and y, it can be written as, $u = x^n f(y/x)$

$$\frac{\partial u}{\partial x} = nx^{n-1} f(y/x) + x^n f'(y/x)\left(\frac{-y}{x^2}\right)$$
$$= nx^{n-1} f(y/x) - y\, x^{n-2} f'(y/x) \quad ...(1)$$

and
$$\frac{\partial u}{\partial y} = x^n f'(y/x)\left(\frac{1}{x}\right)$$
$$= x^{n-1} f'(y/x) \quad ...(2)$$

Multiplying (1) and (2) by x and y respectively and adding, we obtain

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f(y/x) = nu \quad ...(3)$$

Hence the theorem.

This theorem can be generalized to homogeneous functions of any number of variables. Thus, if $u = f(x_1, x_2, ... x_m)$ be a function of degree n of variables $x_1, x_2, ... x_m$, then

$$x_1\frac{\partial u}{\partial x} + x_2\frac{\partial u}{\partial x_2} + x_3\frac{\partial u}{\partial x_3} + ... + x_m\frac{\partial u}{\partial x_m} = nu$$

The proof is similar to that for two variables.

Deduction I: If u is a homogeneous function of x and y of degree n and $u = f(v)$, then

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\frac{f(v)}{f'(v)}, \text{ and}$$

Deduction II: $x^2\dfrac{\partial^2 v}{\partial x^2} + 2xy\dfrac{\partial^2 v}{\partial x\partial y} + y^2\dfrac{\partial^2 v}{\partial y^2} = g(v)[g'(v) - 1]$ where $g(v) = n\dfrac{f(v)}{f'(v)}$.

SOME SOLVED EXAMPLES

Example 4.29. *If* $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *then show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

Solution. Here
$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$\therefore$
$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x, y) \quad \text{(say)} = \sqrt{x}\left[\frac{1+\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}\right]$$

$\Rightarrow$ sin u is a homogeneous function of degree $\dfrac{1}{2}$

Hence by Euler's theorem

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

or
$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}=\frac{1}{2}\tan u. \quad \textbf{Proved.}$$

Example 4.30. *If* $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, *then prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0.$

Solution. Hint: Take $u_1 = \sin^{-1}\left(\frac{x}{y}\right)$ and $u_2 = \tan^{-1}\left(\frac{y}{x}\right)$

Example 4.31. *If* $v = \log_e\left(\frac{x^4+y^4}{x+y}\right)$, *then show that* $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = 3.$

Solution. Given, $$v = \log_e\left(\frac{x^4+y^4}{x+y}\right)$$

Here v is not a homogeneous function, but
$$z = e^v = \frac{x^4+y^4}{x+y}=\frac{x^4(1+y^4/x^4)}{x(1+y/x)}$$
$$= x^3\phi\,(y/x), \quad \Rightarrow \quad z \text{ is a homogeneous functions of degree 3.}$$
Now apply Euler's theorem and prove the same by putting $z = e^v$.

Example 4.32. *If* $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, *show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -3\tan u.$

Solution. Given, $$u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right),$$

$\therefore$ $$\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

Here $\sin u$ is a homogeneous function of degree $1 - 4 = -3$. Hence by Euler's theorem
$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = -3\sin u$$

$\Rightarrow$ $$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}+z\cos u\frac{\partial u}{\partial z} = -3\sin u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u} = -3\tan u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} + 3\tan u = 0 \quad \textbf{Proved.}$$

Example 4.33. *If* $v = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, *then show that*

i. $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = \sin 2v$

ii. $x\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2} = 2\cos 3v\sin v$

Solution. Here v is not a homogeneous function

But $$z = \tan v = \frac{x^3 + y^3}{x - y} = \frac{x^3(1 + y^3/x^3)}{x(1 - y/x)} = f(x) \text{ (say)} = x^2\phi\,(y/x)$$

Therefore z is a homogeneous function of x, y of degree 2. Hence by Euler's deduction-I formula,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\frac{f(v)}{f'(v)} = 2\frac{\tan v}{\sec^2 v} = \frac{2\sin v\cos^2 v}{\cos v}$$
$$= 2\sin v\cos v = \sin 2v$$

Also by Euler's deduction II formula

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = g(v)\,[g'(v) - 1]$$

Here $$g(v) = \sin 2v$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \sin 2v\,(2\cos 2v - 1)$$
$$= 2\sin 2v\cos 2v - \sin 2v$$
$$= \sin 4v - \sin 2v \qquad \left[\because\ \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right.$$
$$= 2\sin v\cos 3v \qquad \textbf{Proved.}$$

Example 4.34. *If* $u = \sin^{-1}\left(\dfrac{x^3 + y^3 + z^3}{ax + by + cz}\right)$, *then prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\tan u.$

Solution. Hint: $\sin u = \dfrac{x^3 + y^3 + z^3}{ax + by + cz} = f(x, y)$ (say)

Here $\sin u$ is a homogeneous function of degree $3 - 1 = 2$, then apply Euler's theorem.

Example 4.35. *If* $u = x\sin^{-1}\left(\dfrac{y}{x}\right)$, *then prove that* $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

Solution. Try yourself.

Example 4.36. *If* $v = \sec^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right)$, *then prove that* $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 2\cot v$, *also evaluate*

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2}.$$

Solution. Hint: $$\sec v = \left(\frac{x^3 + y^3}{x + y}\right) = f(x, y)$$

Example 4.37. *If* $u = \sin^{-1}\left[\dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}\right]^{1/2}$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\,(13 + \tan^2 u).$$

Solution. Let $z = \sin u = \left[\dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}\right]^{1/2}$, since z is a homogeneous function of x and y of degree $-1/12$ *i.e.*, $n = -1/12$.

Now applying Euler's formula, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{12} \tan u, \text{ we have}$$

Again let $g(u) = \dfrac{-1}{12} \tan u$, then

Applying Euler's IInd deduction formula, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\ [g'(u) - 1]$$

$$= \frac{-1}{12} \tan u \left[\frac{-1}{12}\sec^2 u - 1\right]$$

$$= \frac{\tan u}{144}\ (\tan^2 u + 13).$$

EXERCISE 4.4

1. Verify Euler's theorem for the following functions:

i. $u = 3x^2yz + 4xy^2z + 5y^4$ ii. $u = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$

iii. $u = x^2 \log\left(\dfrac{y}{x}\right)$ iv. $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ v. $u = axy + byz + czx$

2. If $u = \dfrac{x^3\,y^3}{x^3 + y^3}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$.

3. If $v = \log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

4. If $u = x^4y^2 \sin^{-1}\left(\dfrac{y}{x}\right)$, then find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

5. If $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

6. If $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

7. If $u = \dfrac{1}{x^3} + \dfrac{1}{x^2 y} + \dfrac{1}{x^3 + 5y^3}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + 3u = 0$.

8. State and Prove Euler's theorem and verify for $u = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$.

Answers

4. $6u$

4.5 TOTAL DIFFERENTIAL COEFFICIENT

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then u itself is a function of a single variable 't', and hence we can find the ordinary derivative $\frac{du}{dt}$. This, in reference to partial differentiation is called the total differentiation of u w.r.t. t.

The problem now is to find $\frac{du}{dt}$ without actually substituting the values of x and y in terms of t in $f(x, y)$. We can proceed as follows:

Let δx, δy and δu denote the small increments in x, y and u respectively corresponding to a small increment δt in t.

We have $\quad x + \delta x = \phi(t + \delta t)$ and $y + \delta y = \psi(t + \delta t)$

$\therefore \quad u + \delta u = f(x + \delta x, y + \delta y)$

Then by definition

$$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y+\delta y) - f(x, y)}{\delta t} \right]$$

$$= \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)}{\delta t} \right]$$

$$= \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} \right] + \lim_{\delta t \to 0} \left[\frac{f(x, y+\delta y - f(x, y))}{\delta y} \frac{\delta y}{\delta t} \right]$$

As $\delta t \to 0$, $\delta x \to 0$ and $\delta y \to 0$

$$\therefore \qquad = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}; \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt};$$

$$\therefore \qquad \frac{du}{dt} = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \right] \cdot \left[\lim_{\delta t \to 0} \frac{\delta x}{\delta t} \right]$$
$$+ \left[\lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \right] \cdot \left[\lim_{\delta t \to 0} \frac{\delta y}{\delta t} \right]$$

But by definition of partial derivatives

$$\lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = \frac{\partial f(x, y)}{\partial y} = \frac{\partial f}{\partial y}$$

and
$$\lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \frac{\partial f(x, y+\delta y)}{\partial x}$$

Now assuming that $\frac{\partial f(x, y)}{\partial x}$ is a continuous function of y,

$$\therefore \qquad \lim_{\delta x \to 0} \frac{\partial f(x, y+\delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x} = \frac{\partial f}{\partial x}$$

Hence $$\frac{du}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

i.e. $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad \text{...(A)}$$

The above result can be extended to a function of more than two variables, *i.e.*,

If $u = f(x_1, x_2, \ldots\ldots x_n)$ and $x_1, x_2, \ldots\ldots, x_n$ are functions of 't' only, then we can prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt} + \ldots\ldots + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt} \qquad \text{...}(B)$$

Deduction. If $u = f(x, y)$, where $x = \phi(t_1, t_2)$ and $y = \psi(t_1, t_2)$, then from the above result,

we have $$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

and $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

Also, from the given equations, we can obtain t_1, t_2 in terms of x, y *i.e.*, $t_1 = \phi_1(x, y)$ and $t_2 = \phi_2(x, y)$

then $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y}$$

By using these equations, we can change one system of independent variables to another system of independent variables.

The higher order partial derivatives of u can be obtained by a repeated application of the above results.

4.6 DIFFERENTIATION OF IMPLICIT FUNCTIONS

The results (A) of the preceding section can be used to find the differential coefficient of implicit functions.

If $u = f(x, y)$ and y is a function of x, then the total differential coefficient of u w.r.t. x is given by

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

In particular, if we are given an implicit relation between x and y of the form, $u = f(x, y) = c$,

where 'c' is a constant, then $$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

i.e., $$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

or, $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

In general, if $u = f(x_1, x_2, x_3 \ldots\ldots x_n)$ and the variables, $x_2, x_3 \ldots\ldots x_n$ are all functions of x_1 then from the result (B) of the preceding section, we obtain

$$\frac{du}{dx_1} = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dx_1} + \frac{\partial u}{\partial x_3}\cdot\frac{dx_3}{dx_1} + \ldots\ldots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dx_1}$$

4.6.1 Second Order Derivative of an Implicit Function

If $f(x, y) = c$, we have $\quad \dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$

Now our aim is to find $\dfrac{d^2y}{dx^2}$ in terms of the partial derivatives of (x, y). For now, we shall denote the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y^2}$ by p, q, r, s and t respectively.

Using these notations, we have

$$\frac{dy}{dx} = -p/q$$

Differentiating this w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-\left(q\dfrac{dp}{dx} - p\dfrac{dq}{dx}\right)}{q^2} \qquad \ldots(1)$$

Since p and q are functions of x and y,

$$\therefore \quad \frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}\cdot\frac{dy}{dx}$$

$$= r + s\left(\frac{-p}{q}\right) = \frac{qr - ps}{q},$$

Similarly, $\quad \dfrac{dq}{dx} = \dfrac{qs - tp}{q}$

Substituting these values in (1), we get

$$\frac{d^2y}{dx^2} = -\left[\frac{q^2r - 2pqs + p^2t}{q^3}\right]$$

SOME SOLVED EXAMPLES

Example 4.38. *If $u = x^5y^4$, where $x = t^2$ and $y = t^3$, find* $\dfrac{du}{dt}$.

Solution. Here u is a function of x and y, where x and y are the functions of t.

$\therefore$ u is also a function of single variable t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (5x^4y^4) \,.\, 2t + (4x^5y^3) \,.\, 3t^2$$
$$= 10x^4y^4t + 12x^5y^3t^2$$
$$= 10t^8 \,.\, t^{12} \,.\, t + 12 \,.\, t^{10} \,.\, t^9 \,.\, t^2$$
$$= 22t^{21}$$

Example 4.39. *If $u = f(x-y, y-z, z-x)$, prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Solution. Hint: $x - y = X$ ...(1) $y - z = Y$ ...(2) $z - x = Z$ ...(3)

$\therefore$ $u = f(X, Y, Z)$, where X, Y, Z, are the functions of x, y, z, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{dY} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{dZ} \cdot \frac{\partial Z}{\partial x} \qquad ...(4)$$

Now find all the required derivatives and put them in (4) to get the required result.

Example 4.40. *If $x = r\cos\theta$, $y = r\sin\theta$, then prove that* $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$.

Solution. Since $x = r\cos\theta, y = r\sin\theta$

$$r^2 = x^2 + y^2 \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \text{or} \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$
$$= \frac{1}{r} - \frac{x}{r^2}\left(\frac{\partial r}{\partial x}\right)$$
$$= \frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right) \text{ (from above)}$$

$$\text{or} \quad \frac{\partial^2 r}{\partial x^2} = \frac{r^2 - x^2}{r^3} \qquad ...(2)$$

Similarly, we can find

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3} \qquad ...(3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3}$$
$$= \frac{2r^2 - (x^2 + y^2)}{r^3} = \frac{2r^2 - r^2}{r^3} = \frac{1}{r}$$

$$\text{Thus, L.H.S.} \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \qquad ...(4)$$

R.H.S. $\frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\} = \frac{1}{r}\left\{\frac{x^2}{r^2}+\frac{y^2}{r^2}\right\}$

$$= \frac{1}{r}\left(\frac{x^2+y^2}{r^2}\right) = \frac{1}{r}\left(\frac{r^2}{r^2}\right) = \frac{1}{r} = \text{L.H.S} \qquad ...(5)$$

Hence by (4) and (5), we get

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\} \quad \textbf{Proved.}$$

Example 4.41. *Find* $\frac{dy}{dx}$ *if* $e^x + e^y = 2xy$.

Solution. Try yourself.

Answer: $\frac{2y-e^x}{e^y-2x}$.

Example 4.42. *If* $f(x, y) = 0$, $\phi(y, z) = 0$, *show that* $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$.

Solution. If $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{-(\partial f/\partial x)}{(\partial f/\partial y)} \qquad ...(1)$$

If $\phi(y, z) = 0$, then

$$\frac{dz}{dy} = -\frac{(\partial \phi/\partial y)}{(\partial \phi/\partial z)} \qquad ...(2)$$

Multiplying (1) and (2), we get

$$\frac{dy}{dx}\cdot\frac{dz}{dy} = \frac{(\partial f/\partial x)}{(\partial f/\partial y)}\cdot\frac{(\partial \phi/\partial y)}{(\partial \phi/\partial z)}$$

or

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} \quad \textbf{Proved}$$

Example 4.43. *If* $u = x \log xy$, *where* $x^3 + y^3 + 3xy = 1$, *find* $\frac{du}{dx}$.

Solution. Try yourself.

Answer: $(1 + \log xy) - \frac{x}{y}\frac{(x^2+y)}{(y^2+x)}$

Example 4.44. *Find* $\frac{dy}{dx}$ *when* $\tan^{-1}\left(\frac{x}{y}\right) + y^3 + 1 = 0$; $x > 0, y > 0$.

Solution. Hint: Use $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

Answer: $\frac{-y}{3y^4+3y^2x^2-x}$

Example 4.45. *If* $x + y = 2\, e^{\theta} \cos \phi$ *and* $x - y = 2\, i\, e^{\theta} \sin \phi$, *show that* $\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial \theta^2} = 4xy\, \dfrac{\partial^2 u}{\partial x \partial y}$.

Solution. We have, $x + y = 2\, e^{\theta} \cos \phi$ ...(1)

and $x - y = 2\, i\, e^{\theta} \sin \phi$ ...(2)

Adding (1) and (2), we get

$$2x = 2\, e^{\theta}\, (i \sin \phi + \cos \phi)$$

$$x = e^{\theta + i\phi} \qquad (\because \quad e^{i\phi} = \cos \phi + i \sin \phi) \qquad ...(3)$$

Subtracting (2) from (1), we get

$$2y = 2e^{\theta}\, (\cos \phi - i \sin \phi) = e^{\theta - i\phi} \qquad ...(4)$$

Clearly, $u = f(x, y)$ and x, y are the functions of θ and ϕ. Hence u is a composite function of θ and ϕ.

Now,
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \theta}$$

$$= x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} \qquad [\because \quad e^{\theta + i\phi} = x,\, e^{\theta - i\phi} = y]$$

and
$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial y}{\partial \theta}$$

or
$$= \frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]x + \frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]y \qquad ...(5)$$

Again,
$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \phi}$$

$$= i\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right]$$

$\therefore$
$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \phi}\right)\left(\frac{\partial x}{\partial \phi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \phi}\right)\frac{\partial y}{\partial \phi}$$

$$= i\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right]\,(ix) + i\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right].\,(-iy)$$

$$= -x\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right]$$

$$= -x\left[\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y}\right] + y\left[x\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\right]$$

or
$$\frac{\partial^2 u}{\partial \phi^2} = -\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] + 2xy\,\frac{\partial^2 u}{\partial x \partial y} - \left[x^2\,\frac{\partial^2 u}{\partial x^2} + y^2\,\frac{\partial^2 u}{\partial y^2}\right] \qquad ...(6)$$

Adding (5) and (6), we get

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\,\frac{\partial^2 u}{\partial x \partial y} \quad \textbf{Proved.}$$

EXERCISE 4.5

1. If $x^x + y^y = c$, find the value of $\dfrac{dy}{dx}$.

2. If $ax^2 + 2hxy + by^2 = 1$, find $\dfrac{d^2y}{dx^2}$.

3. If $u = e^{mx}(y - z)$, $y = m\sin x$ and $z = \cos x$, find $\dfrac{du}{dx}$.

4. If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$, find the values of $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$.

5. If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch each other, show that at the point of contact
$$\frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}.\frac{\partial \phi}{\partial x} = 0$$

6. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

   a. $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$

   b. $\dfrac{\partial^2 r}{\partial x^2}.\dfrac{\partial^2 r}{\partial y^2} = \left(\dfrac{\partial^2 r}{\partial x \partial y}\right)^2$

7. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$, $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$ and $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0$

8. If $z = z(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that $\dfrac{\partial z}{\partial u}.\dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$.

9. If $x^4 + y^4 = 4a^2xy$, prove that $(a^2x - y^3)^3 \dfrac{d^2y}{dx^2} = 2a^2xy\,(3a^4 + x^2y^2)$.

10. If $u = x^2 + y^2 + z^2 - 2xyz = 1$, Prove that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0.$

    Hint: $du = \dfrac{\partial u}{\partial x}\,dx + \dfrac{\partial u}{\partial y}\,dy + \dfrac{\partial u}{\partial z}\,dz = 0.$

11. If $z = f(x, y)$, where $x = e^u\cos v$, $y = e^u\sin v$, prove that; $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = e^{2u}\left[\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2\right].$

12. Transform the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ into polar co-ordinates.

Answers

1. $-\left[\dfrac{y^x.\log_e y + yx^{y-1}}{x^y\log_e x + xy^{x-1}}\right]$

2. $-\dfrac{[(hx+by)^2.a - 2(hx+by)(ax+by)h + (ax+hy)^2 b]}{(hx+by)^2}$

3. $e^{mx}(m^2+1)\sin x$

4. $\dfrac{dy}{dx} = \dfrac{clz - anx}{bny - cmz}, \dfrac{dz}{dx} = \dfrac{amx - bly}{bny - cmz}$

12. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$.

INTERESTING FACTS

- "Total derivative" is sometimes also used as a synonym for the material derivative in fluid mechanics.
- Derivative is a way to show instantaneous rate of change at any given point.
- ***Daffy nition*** is the mathematical operation on any function $f(x)$, and its symbol is d/dx.
- How much does the trachea (windpipe) contract to expel air at the maximum speed during a cough, it is calculated using the concept of derivatives.
- What should be the branching angle, at which blood vessels minimize the energy loss due to friction as blood flows through the branches, is even based on the similar concept.

VIDEO REFERENCES

Continuity of Function of Two Variable

REAL LIFE EXAMPLES

- In satellites and astronomy, there is one technique called Kalman Filter, which is the optimum way to modify a prediction based on observations (lets say any observation in space having noise). In the middle of the ***Kalman filter*** there is a matrix object called a Jacobian, which is the matrix of first order partial derivatives of a vector valued function with respect to its parameters. ***Kalman filters*** are used all over the place, for example in guidance systems, and in finding the path of an autonomous vehicle from sensor data.
- Derivatives can even used in finding profit and loss in business using graphs.
- Derivatives are even used in Linear Approximation.
- Total differential equations can be solved using total derivative.
- The total derivative approximates the function with respect to all of its arguments, not just a single one, and is used when function has several variables.
- The concept of derivatives is used in many ways such as change of temperature or rate of change of shapes and sizes of an object depending on the conditions.

4.7 TAYLOR'S THEOREM FOR TWO VARIABLES

Let $f(x, y)$ be a function of two independent variables x and y and has continuous partial derivatives at all points (x, y), then Taylor's series in two variables is as follows:

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

$$+ \ldots\ldots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y) + \ldots\ldots$$

Proof: Assume that $f(x+h, y+k)$ is a function of single variable, say x, then by Taylor's theorem for functions of single variable, we have

$$f(x+h, y+k) = f(x, y+k) + h\,\frac{\partial}{\partial x}\,f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\,f(x, y+k) + \ldots\ldots$$

Expanding each term,

$$f(x+h, y+k) = \left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\ldots\ldots\right]$$
$$+\ h\frac{\partial}{\partial x}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\ldots\ldots\right]$$
$$+\ \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\ldots\ldots\right]$$

Hence, $$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,.\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots\ldots$$

This is sometimes symbolically written as,

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x, y) + \ldots\ldots$$
$$+\ \frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y) + \ldots\ldots$$

where $$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f = h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}$$

and $$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f = h^n\frac{\partial^n f}{\partial x^n}+nh^{n-1}k\frac{\partial^n f}{\partial x^{n-1}\partial y} + \frac{n(n-1)}{1.2}h^{n-2}k^2\frac{\partial^n f}{\partial x^{n-2}\partial y^2} + \ldots\ldots$$
$$+\ \frac{k^n\partial^n f}{\partial y^n}$$

Note: The above theorem can be easily extended to any number of independent variables.

Another form

$$f(x, y) = f(a, b) + \left[(x-a)\left(\frac{\partial f}{\partial x}\right)_{(a,b)}+(y-b)\left(\frac{\partial f}{\partial y}\right)_{(a,b)}\right]+\frac{1}{2!}$$
$$\times\left[(x-a)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}+2(x-a)(y-b)\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(a,b)}+(y-b)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}\right]+\ldots\ldots$$

If $a = 0, b = 0$, then

$$f(x, y) = f(0, 0) + \left[x\left(\frac{\partial f}{\partial x}\right)_{(0,0)}+y\left(\frac{\partial f}{\partial y}\right)_{(0,0)}\right]+\frac{1}{2!}\left[x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}\right]_{(0,0)} + \ldots\ldots$$

This is called Maclaurin's series for two variables.

SOME SOLVED EXAMPLES

Example 4.46. *Expand e^{xy} about the point (1, 1).*

Solution. Let

$$f(x, y) = e^{xy} \quad \Rightarrow \quad f(1, 1) = e^1 = e$$

$$\frac{\partial f}{\partial x} = y\, e^{xy} \quad \Rightarrow \quad \left(\frac{\partial f}{\partial x}\right)_{(1,1)} = e^1 = e$$

$$\frac{\partial f}{\partial y} = x\, e^{xy} \quad \Rightarrow \quad \left(\frac{\partial f}{\partial y}\right)_{(1,1)} = e^1 = e$$

$$\frac{\partial^2 f}{\partial x^2} = y^2\, e^{xy} \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)} = 1^2 . e^1 = e$$

$$\frac{\partial^2 f}{\partial x \partial y} = xy\, e^{xy} + e^{xy} \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,1)} = 1 . 1\, e^1 + e^1 = 2e$$

$$\frac{\partial^2 f}{\partial y^2} = x^2\, e^{xy} \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)} = 1^2 . e^1 = e$$

By Taylor's theorem for function of two variables, we have

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} f(a,b) + (y-b)\frac{\partial}{\partial y} f(a,b)\right] + \frac{1}{2!}$$
$$\left[(x-a)^2 \frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2 \frac{\partial^2 f}{\partial y^2}(a,b)\right] + \ldots\ldots$$

$$\Rightarrow \quad e^{xy} = e[1 + (x-1) + (y-1)] + \frac{1}{2!}\,[(x-1)^2 + 4(x-1)\,(y-1) + (y-1)^2]e + \ldots\ldots$$

Example 4.47. *Expand $e^x \cos y$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's series.*

Solution. Try yourself.

Answer:
$$\frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)\right] +$$
$$\frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \ldots\ldots$$

Example 4.48. *Find the expansion of cos x cos y in powers of x, y up to fourth order terms.*

Solution. Let

$$f(x, y) = \cos x \cos y \quad \Rightarrow \quad f(0, 0) = \cos 0 \cos 0 = 1$$

$$\frac{\partial f}{\partial x} = -\sin x \cos y \quad \Rightarrow \quad \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y} = -\cos x \sin y \quad \Rightarrow \quad \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x^2} = -\cos x \cos y \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = -1$$

$$\frac{\partial^2 f}{\partial x \partial y} = \sin x \sin y \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = -\cos x \cos y \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = -1$$

$$\frac{\partial^3 f}{\partial x^3} = \sin x \cos y \quad \Rightarrow \quad \left(\frac{\partial^3 f}{\partial x^3}\right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = -\cos x \sin y \quad \Rightarrow \quad \left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \sin x \cos y \quad \Rightarrow \quad \left(\frac{\partial^3 f}{\partial x \partial y^2}\right)_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial y^3} = \cos x \sin y \quad \Rightarrow \quad \left(\frac{\partial^3 f}{\partial y^3}\right)_{(0,0)} = 0$$

$$\frac{\partial^4 f}{\partial x^4} = \cos x \cos y \quad \Rightarrow \quad \left(\frac{\partial^4 f}{\partial x^4}\right)_{(0,0)} = 1$$

$$\frac{\partial^4 f}{\partial x^3 \partial y} = -\sin x \sin y \quad \Rightarrow \quad \left(\frac{\partial^4 f}{\partial x^3 \partial y}\right)_{(0,0)} = 0$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \cos x \cos y \quad \Rightarrow \quad \left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)_{(0,0)} = 1$$

$$\frac{\partial^4 f}{\partial x \partial y^3} = -\sin x \sin y \quad \Rightarrow \quad \left(\frac{\partial^4 f}{\partial x \partial y^3}\right)_{(0,0)} = 0$$

$$\frac{\partial^4 f}{\partial y^4} = \cos x \cos y \quad \Rightarrow \quad \left(\frac{\partial^4 f}{\partial y^4}\right)_{(0,0)} = 1$$

Putting these values in Taylor's theorem, we get

$$f(x, y) = f(0,0) + \left[x \frac{\partial f}{\partial x}(0,0) + y \frac{\partial f}{\partial y}(0,0)\right] + \frac{1}{2!}\left[x^2 \frac{\partial^2 f}{\partial x^2}(0,0) + 2xy \frac{\partial^2 f}{\partial y \partial x}(0,0) + y^2 \frac{\partial^2 f}{\partial y^2}(0,0)\right]$$

$$+ \frac{1}{3!}\left[x^3 \frac{\partial^3 f}{\partial x^3}(0,0) + 3x^2 y \frac{\partial^3 f}{\partial x^2 \partial y}(0,0) + 3xy^2 \frac{\partial^3 f}{\partial x \partial y^2}(0,0) + y^3 \frac{\partial^3 f}{\partial y^3}(0,0)\right] + \frac{1}{4!}$$

$$\times \left[x^4 \frac{\partial^4 f}{\partial x^4}(0,0) + 4xy^3\left(\frac{\partial^4 f}{\partial x \partial y^3}(0,0)\right) + 4x^3 y \frac{\partial^4 f}{\partial x^3 \partial y}(0,0)\right.$$

$$\left. + 6x^2 y^2 \frac{\partial^4 f}{\partial x^2 \partial y^2}(0,0) + y^4 \frac{\partial^4 f}{\partial y^4}(0,0)\right] + \ldots\ldots$$

$$\Rightarrow \cos x \cos y = 1 + 0 + 0 + \frac{1}{2}(-x^2 + 0 - y^2) + \frac{1}{6}(0 + 0 + 0 + 0)$$

$$+ \frac{1}{24}(x^4 + 0 + 6x^2y^2 + 0 + y^4) + \ldots\ldots$$

$$\text{or} \quad \cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \ldots\ldots$$

Example 4.49. *Obtain Taylor's expansion of $f(x, y) = \tan^{-1} y/x$ about (1,1) up to and including the second degree terms. Hence compute $f(1.1, 0.9)$.*

Solution. Try yourself.

Answer: 0.7862.

Example 4.50. *Find the quadratic Taylor's series expansion of $f(x, y) = 2x^3 + 2y^3 - 4x^2y$ about the point (1, 2).*

Solution. Try yourself.

Answer: $10 - 10(x-1) + 20(y-2) - 2(x-1)^2 - 8(x-1)(y-2) + 12(y-2)^2 + 2(x-1)^3 - 4(x-1)^2(y-2) + 2(y-2)^3 + \ldots\ldots$ which can be verified by direct expansion also.

Example 4.51. *Expand $\dfrac{(x+h)(y+k)}{(x+h+y+k)}$ in powers of h and k upto and inclusive of 2nd degree term.*

Solution. Try yourself.

Answer: $\dfrac{xy}{(x+y)} + \dfrac{hy^2}{(x+y)^2} + \dfrac{kx^2}{(x+y)^2} - \dfrac{h^2y^2}{(x+y)^3} + \dfrac{2hkxy}{(x+y)^3} - \dfrac{k^2x^2}{(x+y)^3} + \ldots\ldots$

Example 4.52. *Expand $f(x, y) = x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ by Taylor's theorem.*

Solution. We know by Taylor's theorem,

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots\ldots \quad \ldots(1)$$

Here, $a + h = x$ and $h = x - 1$ so, $a = 1$ and $b + k = y$ and $k = y + 2$ so $b = -2$

$$f(x, y) = x^2y + 3y - 2 \quad \Rightarrow \quad f(1, -2) = -10$$

$$\frac{\partial f}{\partial x} = 2xy \quad \Rightarrow \quad \left(\frac{\partial f}{\partial x}\right)_{(1,-2)} = -4$$

$$\frac{\partial f}{\partial y} = x^2 + 3 \quad \Rightarrow \quad \left(\frac{\partial f}{\partial y}\right)_{(1,-2)} = 4$$

$$\frac{\partial^2 f}{\partial x^2} = 2y \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,-2)} = -4$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,-2)} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 0 \qquad \Rightarrow \qquad \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,-2)} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = 0 \qquad \Rightarrow \qquad \left(\frac{\partial^3 f}{\partial x^3}\right)_{(1,-2)} = 0$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = 2 \qquad \Rightarrow \qquad \left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{(1,-2)} = 2$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = 0 \qquad \Rightarrow \qquad \left(\frac{\partial^3 f}{\partial x \partial y^2}\right)_{(1,-2)} = 0$$

$$\frac{\partial^3 f}{\partial y^3} = 0 \qquad \Rightarrow \qquad \left(\frac{\partial^3 f}{\partial y^3}\right)_{(1,-2)} = 0$$

Putting these values in (1), we get

$$x^2y + 3y - 2 = (-10) + [(x-1)(-4) + (y+2).4]$$
$$+ \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2).2 + (y+2)^2.0]$$
$$+ \frac{1}{3!}[(x-1)^3.0 + 3(x-1)^2(y+2).2 + 0 + 0]$$

or $\quad x^2 + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$

Example 4.53. *Expand x^y in powers of $(x-1)$ and $(y-1)$ upto third degree term.*

Solution. We have $\quad f(x, y) = x^y \qquad$ ...(1)

Here, $\qquad a + h = x$, and $h = x - 1 \Rightarrow a = 1$

and $\qquad b + k = y$, and $k = y - 1 \Rightarrow b = 1$

Also $\qquad f(1, 1) = 1$

Now differentiating (1) w.r.t. x, we get

$$\frac{\partial f}{\partial x} = yx^{y-1} \qquad \Rightarrow \qquad \left(\frac{\partial f}{\partial x}\right)_{(1,1)} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2} \qquad \Rightarrow \qquad \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-3} \qquad \Rightarrow \qquad \left(\frac{\partial^3 f}{\partial x^3}\right)_{(1,1)} = 0$$

Again

$$\frac{\partial f}{\partial y} = x^y \log x \qquad \Rightarrow \qquad \left(\frac{\partial f}{\partial y}\right)_{(1,1)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = x^y (\log x)^2 \qquad \Rightarrow \qquad \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)} = 0$$

$$\frac{\partial^3 f}{\partial y^3} = x^y (\log x)^3 \qquad \Rightarrow \qquad \left(\frac{\partial^3 f}{\partial y^3}\right)_{(1,1)} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^y \cdot \frac{1}{x} + yx^{y-1} \log x = x^{y-1} + yx^{y-1} \log x \quad \Rightarrow \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{x=1,\, y=1} = 1$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = (y-1)\, x^{y-2} + y\,(y-1)\, x^{y-2} \log x + y\, x^{y-2} \quad \Rightarrow \quad \left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{(1,1)} = 1$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = y\, x^{y-1} (\log x)^2 + 2x^{y-1} \log x \quad \Rightarrow \quad \left(\frac{\partial^3 f}{\partial x \partial y^2}\right)_{(1,1)} = 0$$

Now applying Taylor's theorem, we get

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left[h^2 \frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x} + k^2 \frac{\partial^2 f}{\partial y^2}\right]$$
$$+ \frac{1}{3!}\left[h^3 \frac{\partial^3 f}{\partial x^3} + 3h^2 k \frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2 \frac{\partial^3 f}{\partial x \partial y^2} + k^3 \frac{\partial^3 f}{\partial y^3}\right] + \ldots$$

$$\therefore \qquad f(x, y) = x^y = 1 + (x-1) + 0 + \frac{1}{2!}\,(0 + 2\,(x-1)\,(y-1) + 0)$$
$$+ \frac{1}{3!}\,[0 + 3\,(x-1)^2 (y-1) + 0 + 0] + \ldots\ldots$$

$$\Rightarrow \qquad x^y = 1 + (x-1) + (x-1)\,(y-1) + \frac{1}{2}\,(x-1)^2\,(y-1).$$

Example 4.54. *Expand $e^x \sin y$ in powers of x and y as far as term of third degree.*

Solution. Let $f(x, y) = e^x \sin y$

Taylor's series in the powers of x and y.

$$f(x, y) = f(0, 0) + xf_x(0, 0) + y f_y(0, 0) + \frac{x^2}{2!} f_{xx}(0, 0) + \frac{y^2}{2!} f_{yy}(0, 0) + \frac{2xy}{2!} f_{xy}(0, 0) +$$
$$+ \frac{x^3}{3!} f_{xxx}(0, 0) + \frac{y^3}{3!} f_{yyy}(0, 0) + \frac{3x^2 y}{3!} f_{xxy}(0, 0) + \frac{3xy^2}{3!} f_{xyy}(0, 0) + \ldots\ldots \qquad \ldots(\underline{A})$$

Now,

$$\begin{aligned}
f(x, y) &= e^x \sin y & \therefore \quad f(0, 0) &= 0\\
f_x(x, y) &= e^x \sin y & \therefore \quad f_x(0, 0) &= 0\\
f_y(x, y) &= e^x \cos y & \therefore \quad f_y(0, 0) &= 1\\
f_{xx}(x, y) &= e^x \sin y & \therefore \quad f_{xx}(0, 0) &= 0\\
f_{xy}(x, y) &= e^x \cos y & \therefore \quad f_{xy}(0, 0) &= 1\\
f_{yy}(x, y) &= -e^x \sin y & \therefore \quad f_{yy}(0, 0) &= 0\\
f_{xxx}(x, y) &= e^x \sin y & \therefore \quad f_{xxx}(0, 0) &= 0\\
f_{xxy}(x, y) &= e^x \cos y & \therefore \quad f_{xxy}(0, 0) &= 1\\
f_{xyy}(x, y) &= -e^x \sin y & \therefore \quad f_{xyy}(0, 0) &= 0\\
f_{yyy}(x, y) &= -e^x \cos y & \therefore \quad f_{yyy}(0, 0) &= -1
\end{aligned}$$

Putting all these values in equation (A), we get

$$f(x, y) = e^x \sin y = 0 + x(0) + y(1) + \frac{x^2}{2!}(0) + x\,y\,(1) + \frac{y^2}{2!}(0)$$

$$+ \frac{x^3}{3!}(0) + \frac{3x^2y}{3!}(1) + \frac{3xy^2}{3!}(0) + \frac{y^3}{3!}(-1) + \ldots$$

Hence $$e^x \sin y = y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \ldots\ldots$$

Example 4.55. *If $f(x, y) = \tan^{-1}(xy)$, compute an approximate value of $f(0.9, -1.2)$.*

Solution. Given $f(x, y) = \tan^{-1}(xy)$ $\quad\therefore\quad$ $f(1, -1) = \dfrac{-\pi}{4}$ ...(*i*)

Let us expand $f(x, y)$ near the point $(1, -1)$.

$$f(0.9, -1.2) = f(1 - 0.1, -1 - 0.2)$$

$$= f(1, -1) + [(0.1)\, f_x(1, -1) + (-0.2)\, f_y(1, -1)]$$

$$+ \frac{1}{2!}[(-0.1)^2 f_{xx}(1, -1) + (-0.2)^2 f_{yy}(1, -1)$$

$$+2(-0.1)(-0.2)\, f_{xy}(1, -1)] + \ldots. \qquad \text{...(A)}$$

From equation (1)

$$f_x(x, y) = \frac{y}{1 + x^2y^2} \quad\therefore\quad f_x(1, -1) = -\frac{1}{2} \qquad \text{...}(ii)$$

$$f_y(x, y) = \frac{x}{1 + x^2y^2} \quad\therefore\quad f_y(1, -1) = \frac{1}{2} \qquad \text{...}(iii)$$

$$f_{xx}(x, y) = \frac{-2xy^3}{(1 + x^2y^2)^2} \quad\therefore\quad f_{xx}(1, -1) = \frac{1}{2} \qquad \text{...}(iv)$$

$$f_{yy}(x, y) = \frac{-2x^3y}{(1 + x^2y^2)^2} \quad\therefore\quad f_{yy}(1, -1) = \frac{1}{2} \qquad \text{...}(v)$$

$$f_{xy}(x, y) = \frac{1 - x^2y^2}{(1 + x^2y^2)^2} \quad\therefore\quad f_{xy}(1, -1) = 0 \qquad \text{...}(vi)$$

Putting all these values in equation (A), we get

$$f(0.9, -1.2) = \frac{-\pi}{4} + (-0.1)\left(-\frac{1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)$$

$$+ \frac{1}{2}\left[(-0.1)^2\left(\frac{1}{2}\right) + 2(-0.1)(-0.2)(0) + (-0.2)^2\left(\frac{1}{2}\right)\right] + \ldots. \qquad \left[\because \frac{\pi}{4} = 0.785\right.$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + \frac{1}{2}[0.005 + 0.04]$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + 0.0225$$

$= -0.8128$ **Answer**

4.8 JACOBIANS

INTRODUCTION

In the study of function of several variables, we find a special function of their derivatives known as functional determinant of the given functions. The functional determinants were first studied by the German mathematician Carl Gustav Jacobi (1804-1851). These functional determinants are called Jacobians, after his name. In this topic we shall deal with these determinants and their properties. An important application of Jacobian is in connection with the change of variables of multiple integrals.

Definition If $u_1, u_2, \ldots\ldots u_n$ are functions of n independent variables $x_1, x_2, \ldots\ldots x_n$, then determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \ldots\ldots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \ldots\ldots & \frac{\partial u_2}{\partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \ldots\ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots\ldots u_n$ with respect to $x_1, x_2, \ldots\ldots x_n$

and is denoted by the symbol $\frac{\partial(u_1, u_2, \ldots\ldots u_n)}{\partial(x_1, x_2, \ldots\ldots x_n)}$. It is also denoted by $J(u_1, u_2, \ldots\ldots u_n)$ or simply by J when there is no ambiguity with regards to the variables.

4.8.1 Properties Of Jacobians

1. Let $J_1 = \frac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \frac{\partial(x, y)}{\partial(u, v)}$, then $J_1 \cdot J_2 = 1$.
2. If u, v are functions of s, t, where s, t are functions of x, y, then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(s, t)} \times \frac{\partial(s, t)}{\partial(x, y)}$$

A SPECIAL FORM

If the function $u_1, u_2, \ldots\ldots\ldots u_n$ of the variables $x_1, x_1, \ldots\ldots x_n$ are defined by the relation $u_1 = f_1(x_1)$; $u_2 = f_2(x_1, x_2)$; ...... $u_n = f_n(x_1, x_2 \ldots\ldots x_n)$

Then, $$\frac{\partial(u_1, u_2, \ldots\ldots u_n)}{\partial(x_1, x_2, \ldots\ldots x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & \ldots\ldots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \ldots\ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \ldots\ldots \frac{\partial u_n}{\partial x_n}.$$

i.e., the Jacobians reduce to the product of the leading diagonal elements of the determinants.

3. If u, v, w are functions of three independent variables x, y, z and functionally related, then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

SOME SOLVED EXAMPLES

Example 4.56. *If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, prove that they are not independent. Find the relation between u, v and w.*

Solution. Here u, v and w are not independent, if $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$ ...(1)

Now
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad ...(2)$$

Here
$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 2, \frac{\partial u}{\partial z} = 1$$
$$\frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial y} = -2, \frac{\partial v}{\partial z} = 3$$
$$\frac{\partial w}{\partial x} = 2y - z, \frac{\partial w}{\partial y} = 2x + 4z, \frac{\partial w}{\partial z} = -x + 4y - 4z$$

From (2), we have,
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$,
$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y - z & 2x - 4y + 6z & -x + 2y - 3z \end{vmatrix}$$
$$= -4(-x + 2y - 3z) - 2(2x - 4y + 6z)$$
$$= 4x - 8y + 12z - 4x + 8y - 12z = 0$$

Hence, u, v, w are not independent

Now
$$u + v = 2x + 4z \quad ...(3)$$
$$u - v = 4y - 2z \quad ...(4)$$

Multiplying (3) and (4), we get
$$(u + v)(u - v) = (2x + 4z)(4y - 2z)$$
$$u^2 - v^2 = 4(2xy + 4yz - zx - 2z^2)$$
$$\Rightarrow \quad u^2 - v^2 = 4w$$

Example 4.57. *Verify whether the following functions are functionally related, and if so, find the relation between them.*

$$u = \frac{x + y}{1 - xy}, \; v = \tan^{-1} x + \tan^{-1} y.$$

Solution. We know that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{(1+x^2)}{(1-xy)^2} \\ \frac{1}{(1+x^2)} & \frac{1}{(1+y^2)} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence, u, v are functionally dependent. Now, we know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

$$\Rightarrow \qquad v = \tan^{-1} u \text{ (As given)} \qquad \therefore \quad u = \tan v$$

Example 4.58. *If* $x = r\cos\theta$, $y = r\sin\theta$, *find* $\frac{\partial(x,y)}{\partial(r,\theta)}$ *and* $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Solution. Here, $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore \qquad \frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial y}{\partial r} = \sin\theta$$

and $$\frac{\partial x}{\partial \theta} = -r\sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta$$

we have $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

Now, $$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}(y/x)$$

$$\therefore \qquad = \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = y/r$$

and $$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} \qquad \Rightarrow \qquad \frac{\partial \theta}{\partial x} = \frac{-y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} \qquad \Rightarrow \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

$$\therefore \qquad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

Remark: $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \dfrac{1}{r} = 1.$

Example 4.59. *If* $x = r\sin\theta \cdot \cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, *show that* $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta.$

Solution. Here, we have,

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

and
$$\frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta, \quad \frac{\partial z}{\partial \phi} = 0$$

Now, we have,
$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi) +$$
$$r\sin\theta\,(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi) \qquad \text{[expanding by 3rd row]}$$
$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$$
$$= r^2\sin\theta$$

Example 4.60. *If* $y_1 = \dfrac{x_2\,x_3}{x_1}$, $y_2 = \dfrac{x_3\,x_1}{x_2}$, $y_3 = \dfrac{x_1\,x_2}{x_3}$ *prove that Jacobian of* y_1, y_2, y_3 *with respect to* x_1, x_2, x_3 *is 4.*

Solution. Try yourself.

Example 4.61. *If* $x = r\cos\theta\cos\phi$, $y = r\sin\theta\sqrt{1-m^2\sin^2\phi}$, $z = r\sin\phi\sqrt{1-n^2\sin^2\theta}$, *where* $m^2 + n^2 = 1$, *find the Jacobian of* $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Solution. Squaring and adding the given relation, we have,

$$x^2 + y^2 + z^2 = r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta\,(1 - m^2\sin^2\phi) + r^2\sin^2\phi\,(1 - n^2\sin^2\theta)$$
$$= r^2(\cos^2\theta\cos^2\phi + \sin^2\theta - \sin^2\theta\sin^2\phi + \sin^2\phi) \qquad (\because \; m^2 + n^2 = 1)$$
$$= r^2(\cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi) \qquad (\because \; 1 - \sin^2\phi = \cos^2\phi)$$
$$= r^2(\cos^2\phi + \sin^2\phi) = r^2$$

Differentiating this partially w.r.t. r, θ, ϕ, we get

$$x\frac{\partial x}{\partial r}+y\frac{\partial y}{\partial r}+z\frac{\partial z}{\partial r}=r \qquad \text{...(1)}$$

$$x\frac{\partial x}{\partial \theta}+y\frac{\partial y}{\partial \theta}+z\frac{\partial z}{\partial \theta}=0 \qquad \text{...(2)}$$

and $$x\frac{\partial x}{\partial \phi}+y\frac{\partial y}{\partial \phi}+z\frac{\partial z}{\partial \phi}=0 \qquad \text{...(3)}$$

$$\therefore \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix}$$

$$=\frac{1}{x}\begin{vmatrix} r & 0 & 0\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix}$$

On adding $(R_2y + R_2z)$ to xR_1 and using (1) to (3)

$$=\frac{r}{x}\begin{vmatrix}\frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix}$$

$$=\frac{r}{x}\begin{vmatrix} r\cos\theta\sqrt{1-m^2\sin^2\phi} & \frac{-r\sin\theta\,.\,m^2\sin\phi\cos\phi}{\sqrt{(1-m^2\sin^2\phi)}}\\ \frac{-r\sin\phi\,.\,n^2\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}} & r\cos\phi\sqrt{1-n^2\sin^2\theta}\end{vmatrix}$$

$$=\frac{r}{x}\left[r^2\cos\theta\cos\phi\sqrt{(1-m^2\sin^2\phi)}\sqrt{(1-n^2\sin^2\theta)}-\right.$$

$$=\left.-\frac{m^2n^2r^2\sin^2\theta\sin^2\phi\cos\theta\cos\phi}{\sqrt{(1-m^2\sin^2\phi)}\sqrt{(1-n^2\sin^2\theta)}}\right]$$

$$=\frac{r^3\cos\theta\cos\phi}{x}\left[\frac{(1-m^2\sin^2\phi)(1-n^2\sin^2\theta)-m^2n^2\sin^2\theta\sin^2\phi}{\sqrt{(1-m^2\sin^2\phi)}\sqrt{(1-n^2\sin^2\theta)}}\right]$$

$$=\frac{r^2(1-m^2\sin^2\phi-n^2\sin^2\theta)}{\sqrt{(1-m^2\sin^2\phi)(1-n^2\sin^2\theta)}} \qquad (\because \quad m^2+n^2=1)$$

Example 4.62. *If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_1 \sin x_2 \cos x_3$, then prove that*

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3.$$

Solution. We have $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$ and $y_3 = \sin x_1 \sin x_2 \cos x_3$,

Now $$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x_1 & 0 & 0 \\ \cos x_1 \cos x_2 & -\sin x_1 \sin x_2 & 0 \\ \cos x_1 \sin x_2 \cos x_3 & \sin x_1 \cos x_2 \cos x_3 & -\sin x_1 \sin x_2 \sin x_3 \end{vmatrix}$$

Expanding R_1, we get, $$= -\sin x_1 [\sin^2 x_1 \sin^2 x_2 \sin x_3]$$

$$= -\sin^3 x_1 \sin^2 x_2 \sin x_3$$

Example 4.63. *In cylindrical co-ordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ prove that* $= \dfrac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$

Solution. We have $\dfrac{\partial x}{\partial \rho} = \cos\phi$, $\dfrac{\partial x}{\partial \phi} = -\rho\sin\phi$, $\dfrac{\partial x}{\partial z} = 0$, $\dfrac{\partial y}{\partial \rho} = \sin\phi$, $\dfrac{\partial y}{\partial \phi} = \rho\cos\phi$, $\dfrac{\partial y}{\partial z} = 0$

and $\dfrac{\partial z}{\partial \rho} = 0$, $\dfrac{\partial z}{\partial \phi} = 0$, $\dfrac{\partial z}{\partial z} = 1$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \rho.$ **Proved.**

EXERCISE 4.6

1. If $u = \dfrac{y^2}{2x}$, $v = \dfrac{x^2 + y^2}{2x}$ find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

2. If $x = a \cos u \cos h\, v$, $y = a \sin u \sin h\, v$, prove that $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{2} a^2 (\cos 2u - \cos h2kv)$.

3. If $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x(x + 2y - z)$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$.

4. If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(u, v)}{\partial(r, \theta)}$.

5. If $u=\dfrac{x}{y-z}, v=\dfrac{y}{z-x}, w=\dfrac{z}{x-y}$, prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=0$.

6. If $x = uv, y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

7. For the transformation $x = e^u \cos v, y = e^u \sin v$, show that $\dfrac{\partial(x,y)}{\partial(u,v)} \times \dfrac{\partial(u,v)}{\partial(x,y)} = 1$.

8. Verify $JJ' = 1$, if $x = e^v \sec u, y = e^v \tan u$.

9. If $u_1 = \dfrac{x_1}{n}, u_2 = \dfrac{x_2}{n}, \ldots u_n = \dfrac{x_n}{n}$ and $x_1^2 + x_2^2 + \ldots\ldots + x_n^2 = 1$, find $\dfrac{\partial(u_1, u_2, \ldots\ldots u_n)}{\partial(x_1, x_2, \ldots\ldots x_n)}$.

10. If $u = \dfrac{y-x}{1+xy}$ and $v = \tan^{-1} y - \tan^{-1} x$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

11. If $x_1 = r \sin\theta_1 \sin\theta_2, x_2 = r \sin\theta_1 \cos\theta_2, x_3 = r\cos\theta_1 \sin\theta_2, x_4 = r\cos\theta_1\cos\theta_2$, show that

$$J(x_1, x_2, x_3, x_4) = r^2 \sin\theta_1 \cos\theta_1$$

12. Find jacobian of $y_1, y_2, \ldots\ldots y_n$ being given

$$y_1 = x_1(1-x_2), y_2 = x_1 x_2 (1-x_3) \ldots\ldots y_{n-1} = x_1 x_2 \ldots\ldots x_{n-1}(1-x_n)$$
$$y_n = x_1 x_2 \ldots\ldots\ldots x_{n-1} x_n$$

13. If $x = u(1-v), y = uv$, verify $JJ' = 1$.

14. Verify that $\dfrac{\partial(x,y)}{\partial(u,v)} \times \dfrac{\partial(u,v)}{\partial(x,y)} = 1$, if $x = \sin\theta\cos\phi, y = \sin\theta\sin\phi$.

15. If $u = x^2 - y^2, v = 2xy$ and $x = r\cos\theta, y = r\sin\theta$, then prove that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Answers

1. $-y/2x$ 4. $-4r^3$ 6. $\dfrac{(u-v)^2}{4uv}$ 9. $\dfrac{1}{x_n^{n-1}}$

10. 0 12. $x_1^{n-1} x_2^{n-2} \ldots\ldots\ldots x_{n-1}$.

4.9 EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES AND LAGRANGE'S METHOD OF MULTIPLIERS

Introduction: In this topic, we shall determine the values of a function which are greatest or least in their immediate neighbourhood, technically they are termed as maximum and minimum values of the function. A knowledge of these values is helpful in tracing of curves and in determining the greatest and least values of a function in any given finite interval.

Maxima and minima of a function: A function $f(x)$ is said to have a maximum and minimum at $x = a$, if there exists a small nbd. of 'a' *i.e.*, if a positive number ε exists, such that $f(x) < f(a)$ or $f(x) > f(a)$ for all values of x for which $0 < |(x-a)| < \varepsilon$.

Note 1. The term extreme values is used for maximum as well as for a minimum value and the points where these values occur are called extreme points.

Note 2. It should also be noted that a maximum or a minimum value $f(a)$ of $f(x)$ is the greatest or the least value of the function in only a small nbd. of the point $x = a$ and not necessarily in the entire interval of definition of $f(x)$. In fact, a function may have several maximum and minimum values.

Stationary value: A function $f(x)$ is said to have a stationary at $x = a$, if the derivative $f'(x)$ vanishes at $x = a$, *i.e.*, if $f'(a) = 0$. The value $f(a)$ is said to be a stationary value or a turning value of $f(x)$. The term 'stationary' arises from the fact that the rate of change, $f'(x)$ of the function $f(x)$ w.r.t. x is zero for a value of x for which $f(x)$ is stationary.

It should be noted that a maximum or a minimum value of a function is also a stationary value but a stationary value may neither be a maximum nor a minimum value. For example, the function $f(x) = (x - 1)^3 + 2$ has neither a maximum nor a minimum at $x = 1$, despite the fact that $f(x)$ has stationary value at this point.

4.9.1 MAXIMA AND MINIMA OF A FUNCTION OF SEVERAL INDEPENDENT VARIABLES

The definition of maximum or minimum value of a function of several independent variables can be given in the same way as is given for the function of a single variable. Thus, the function $f(x, y, z, \ldots\ldots)$ is said to have a maximum or minimum value at the point $(a, b, c \ldots\ldots)$ if $f(x, y, z, \ldots\ldots) < f(a, b, \ldots\ldots)$ or $f(x, y, z, \ldots\ldots) > f(a, b, \ldots\ldots)$ for all points $(x, y, z, \ldots\ldots)$ in the neighbourhood of the point $(a, b, c, \ldots\ldots)$.

4.9.1.1 Maxima and Minima of a Function of Two Independent Variables

A function $f(x, y)$ of two independent variables is said to have maximum (local maximum) at the point (a, b) if there exists a small positive number δ such that $f(a + h, b + k) < f(a, b)$ for all values of h, k such that $|h| < \delta, |k| < \delta$.

A function $f(x, y)$ is said to have minimum (local minimum) at (a, b) if there exists a small positive number δ such that $f(a + h, b + k) > f(a, b)$ for all h, k such that $|h| < \delta$ and $|k| < \delta$.

We now establish conditions for existence of maximum and minimum of a function of two independent variables.

Necessary Conditions for the Existence of Maxima and Minima

Let $f(a, b)$ be an extreme (a maximum or a minimum) value of the function $f(x, y)$ at the point (a, b). Then, by definition $f(a + h, b + k) - f(a, b)$ should preserve a fixed sign (positive for minimum and negative for maximum) for all sufficiently small values of h and k.

Now, from Taylor's series expansion, we have,

$$f(a + h, b + k) - f(a, b) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}} + \ldots\ldots \quad \ldots(1)$$

By taking h, k sufficiently small, the first degree terms in h, k can be made to govern the sign of the R.H.S. and therefore of the L.H.S. of (1). But the first degree terms change their sign when the signs of h and k are reversed. Hence in order that L.H.S. may preserve a fixed sign, it is necessary that

$$\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} = 0 \quad \ldots(2)$$

Since h and k are arbitrary and independent of each other, we must have

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} = 0$$

Hence, the necessary condition for $f(x, y)$ to have a maximum or minimum value at (a, b) are that, the first order partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ must vanish for $x = a$ and $y = b$.

Sufficient Conditions for the Existence of Maxima and Minima

If for the function $f(x, y)$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, both vanish at $x = a$ and $y = b$, then from (1), we have

$$f(a + h, b + k) - f(a, b) = \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}} + \ldots\ldots$$

$$= \frac{1}{2!}(rh^2 + 2shk + tk^2) + \ldots\ldots \qquad \ldots(3)$$

where, for brevity, we have put

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}, s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}}, t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$$

for the sufficiently small values of h and k.

The sign of R.H.S. in (3) is governed by the second degree terms in h, k and therefore of the L.H.S. Thus, if for all possible values of h, k, the sign of $(rh^2 + 2shk + tk^2)$ is positive, we have a minima of $f(x, y)$ and if it is negative, we have a maxima. To identify the sign of expression $(rh^2 + 2\,h\,k\,s + t\,k^2)$, we write this as,

$$r\,h^2 + 2\,hks + tk^2 = \frac{1}{r}\,(r^2h^2 + 2rshk + rt\,k^2)$$

$$= \frac{1}{r}\,[(rh + sk^2) + (rt - s^2)\,k^2] \qquad \ldots(4)$$

We have now following cases:

1. If $r > 0$ and $(rt - s^2) > 0$, the sign of (4) is positive.
2. If $r < 0$ and $(rt - s^2) > 0$, the sign of (4) is negative.
3. If $(rt - s^2) < 0$, the sign of (4) cannot be identified.
4. If $(rt - s^2) = 0$ and $rh + sk \neq 0$, the sign of (4) is positive and negative according as r is positive or negative, however, if $rt - s^2 = 0$ and $rh + sk = 0$, further investigations are required.

Hence, if $r > 0$ and $rt - s^2 > 0$, the function $f(x, y)$ has a minimum at (a, b). If $r < 0$ and $rt - s^2 > 0$, the function $f(x, y)$ has a maximum at (a, b). However, if $rt - s^2 < 0$, the function has neither *a* maximum nor a minimum at (a, b).

Again, if $rt - s^2 = 0$ and $rh + sk \neq 0$, the sign of r only will decide the maxima and minima of the function.

If $rt - s^2 = 0$ and $rh + sk = 0$, then second degree terms in (4) vanish and we must consider the terms of higher order in h and k,

The above conditions are sufficient conditions (also called Lagrange's conditions) to determine a maximum and a minimum of a function.

Working Rule: In order to determine a maxima or a minima of a function $f(x, y)$, we proceed as follows:

1. Find $\dfrac{\partial f}{\partial x}$, and $\dfrac{\partial f}{\partial y}$ equate them to zero. Solve these simultaneous equations for x and y. Let the roots be (a_i, b_i), $i = 1, 2, 3, \ldots\ldots$. These (a_i, b_i) are called stationary points.
2. Find $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}$, and $\dfrac{\partial^2 f}{\partial y^2}$ substitute $x = a_i; y = b_i$ in turn. calculate the value $rt - s^2$ for each pair of values.
3. If $r > 0$ and $rt - s^2 > 0$ for a pair of roots, $f(x, y)$ is a minimum for this pair. If $r < 0$ and $rt - s^2 > 0$, it is a maximum. If $rt - s^2 < 0$, the function has neither a maximum nor a minimum and the point is called saddle point. If $rt - s^2 = 0$, the case is undecided and further investigations are necessary to decide the nature of the given function.

SOME SOLVED EXAMPLES

Example 4.64. *Find the maximum and minimum values of the function* $(x - 1)(x - 2)(x - 3)$.

Solution. Let
$$f(x) = (x - 1)(x - 2)(x - 3)$$
$$= x^3 - 6x^2 + 11x - 6$$

then for maximum or minimum, $f'(x) = 0$, *i.e.*,

$$\Rightarrow \quad 3x^2 - 12x + 11 = 0$$

$$\Rightarrow \quad x = 2 + \frac{1}{\sqrt{3}},\ 2 - \frac{1}{\sqrt{3}}$$

Also
$$f''(x) = 6x - 12$$

Clearly $f''\left(2 + \dfrac{1}{\sqrt{3}}\right)$ is positive and $f''\left(2 - \dfrac{1}{\sqrt{3}}\right)$ is negative.

Hence, the given function has maximum at $2 - \dfrac{1}{\sqrt{3}}$ and a minimum at $2 + \dfrac{1}{\sqrt{3}}$. The maximum value of $f(x)$ is, therefore

$$\left(1 - \frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\left(-1 - \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$$

and the minimum value is $\left(1 + \dfrac{1}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{3}}\right)\left(-1 + \dfrac{1}{\sqrt{3}}\right) = -\dfrac{2}{3\sqrt{3}}$.

Example 4.65. *A gas holder is a cylindrical vessel closed at the top and open at the bottom (which dips into water), what should be the ratio of the height to the diameter in order that for a given value its construction may require the least amount of material?*

Solution. Let R be the radius of cylindrical vessel and d be the depth of tank.

Let V be the given capacity of the tank and A be the area of the sheet required to construct it, then

$$V = \pi R^2 d \qquad \ldots(1)$$

$$\therefore \quad d = \frac{V}{\pi R^2}$$

and $\quad A$ = Area of base + Area of cylindrical wall

$\Rightarrow \quad A = \pi R^2 + 2\pi Rd$

or $\quad A = \pi R^2 + 2\pi R \,.\, \dfrac{V}{\pi R^2}$ [from (1)]

$\Rightarrow \quad A = \pi R^2 + \dfrac{2V}{R} \quad \Rightarrow \quad \dfrac{dA}{dR} = 2\pi R - \dfrac{2V}{R^2}$

Again $\quad \dfrac{d^2A}{dR^2} = 2\pi + \dfrac{4V}{R^3}$

For maximum or minimum, $\dfrac{dA}{dR} = 0$

$$\Rightarrow \quad 2\pi R - \frac{2V}{R^2} = 0 \quad \Rightarrow \quad V = \pi R^3$$

$$\left(\frac{d^2A}{dR^2}\right)_{V=\pi R^3} = 6\pi > 0, \text{ (minimum)}$$

Hence $\quad d = \dfrac{V}{\pi R^2} \Rightarrow d = \dfrac{\pi R^3}{\pi R^2} = R$

$$\Rightarrow \quad \frac{d}{2R} = \frac{1}{2} \quad \Rightarrow \quad \frac{\text{Height}}{\text{Diameter}} = \frac{d}{D} = \frac{1}{2}.$$

Example 4.66. *Assuming, that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity. Find the most economical speed when going against a current of c km/hour.*

Solution. Let the velocity of the motor boat be v km./hour. The velocity of the boat relative to the current $(v - c)$ km./hour. Let the total distance to be covered be d km. Then time taken $\dfrac{d}{v-c}$ hours.

Also, we are given that the petrol burnt in one hour $= \alpha v^3$, where α is constant of proportionality.

$\therefore$ If P be the total amount of petrol burnt in covering d kms., then

$$P = \frac{\alpha v^3 d}{v - c} \quad \text{...(1)}$$

$$\therefore \quad \frac{dP}{dv} = \alpha d \,.\, \left[\frac{(v-c)\,.\,3v^2 - v^3\,.\,1}{(v-c)^2}\right] = \frac{\alpha d(2v^3 - 3cv^2)}{(v-c)^2}$$

and differentiating, we have

$$\frac{d^2P}{dv^2} = \frac{\alpha d}{(v-c)^4}\,[(v-c)^2\,(6v^2 - 6vc) - (2v^3 - 3cv^2)\,.\,2(v-c)]$$

$$= \frac{\alpha d}{(v-c)^3}\,[(6v\,.\,(v-c)^2 - 2v^2\,(2v - 3c)]$$

for maximum or minimum, $\dfrac{dP}{dv} = 0$

$$2v^3 - 3cv^2 = 0 \quad \Rightarrow \quad v = 0 \text{ or } \frac{3}{2}\,c$$

If $v = 0$, $P = 0$, *i.e.*, no petrol is burnt.

If $V = \dfrac{3c}{2}, \dfrac{d^2P}{dv^2} = +\text{ve}$ *i.e.*, P is minimum.

$\therefore$ Petrol burnt is minimum when $v = \dfrac{3c}{2}$, *i.e.*, the most economical speed is $\left(\dfrac{3c}{2}\right)$ km/hour.

Example 4.67. *Find the maximum or minimum values of the function* $x^3y^2\,(1 - x - y)$.

Solution. Try yourself.

Answer: Function has a maximum at $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$ and the maximum value is $\dfrac{1}{432}$.

Example 4.68. *Find a point within triangle such that the sum of the squares of its distances from the three vertices, is minimum.*

Solution. Let (x_n, y_n); $n = 1, 2, 3$ be the vertices of the triangle and let (x, y) be any point inside the triangle.

Then let
$$v = \sum_{n=1}^{3} [(x - x_n)^2 + (y - y_n)^2] \qquad \text{...(1)}$$

$\therefore$
$$\frac{\partial v}{\partial x} = \sum_{n=1}^{3} 2(x - x_n) = 2[(x - x_1) + (x - x_2) + (x - x_3)]$$

and
$$\frac{\partial v}{\partial y} = \sum_{n=1}^{3} 2(y - y_n) = 2[(y - y_1) + (y - y_2) + (y - y_3)]$$

and also
$$r = \frac{\partial^2 v}{\partial x^2} = 6,\ s = \frac{\partial^2 v}{\partial x \partial y} = 0 \text{ and } t = \frac{\partial^2 v}{\partial y^2} = 6$$

For a maximum or minimum of v, we have
$$\frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad 2[(x - x_1) + (x - x_2) + (x - x_3)] = 0$$

$$\Rightarrow \quad x = \frac{x_1 + x_2 + x_3}{3}$$

and
$$\frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad 2[(y - y_1) + (y - y_2) + (y - y_3)]$$

$$\Rightarrow \quad y = \frac{y_1 + y_2 + y_3}{3}$$

Thus $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$ is the only point at which v may have a maximum or minimum.

At this point $r = 6$, $s = 0$, $t = 6$, so that $rt - s^2 = 36$ (+ve)

Hence the stationary value of v at the point $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$ is an extreme value.

But $r > 0$, then this extreme value of v is a minimum.

Thus the required point at which v is minimum is $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$. This is known as centroid of the triangle.

Example 4.69. *Find the stationary point of the function* $z = x^3y^2(12 - x - y)$ *satisfying the condition* $x > 0, y > 0$ *and examine their nature.*

Solution. Let $z = x^3y^2(12 - x - y)$ ...(1)

$\therefore$ $$\frac{\partial z}{\partial x} = x^3y^2(-1) + (12 - x - y)y^2 \,.\, 3x^2$$
$$= 3x^2y^2(12 - x - y) - x^3\,y^2$$

$\therefore$ $$\frac{\partial z}{\partial x} = 36x^2y^2 - 4x^3y^2 - 3x^2y^3 \qquad ...(2)$$

Also $$\frac{\partial z}{\partial y} = x^3y^2\,(-1) + (12 - x - y)x^3.2y$$
$$= 24x^3y - 2x^4y - 2x^3y^2 - x^3y^2$$

or $$\frac{\partial z}{\partial y} = 24x^3y - 2x^4y - 3x^3y^2 \qquad ...(3)$$

For maximum or minimum value of z, we have
$$\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial y} = 0$$

$\therefore$ $$\frac{\partial z}{\partial x} = 0 \Rightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$\Rightarrow$ $x^2y^2(36 - 4x - 3y) = 0$

$\Rightarrow$ $$x = 0, y = 0, 4x + 3y = 36 \qquad ...(4)$$

Also $$\frac{\partial z}{\partial y} = 24x^3y - 2x^4y - 3x^3y^2 = 0$$

or $x^3y(24 - 2x - 3y) = 0$

$\Rightarrow$ $$x = 0, y = 0, 2x + 3y = 24 \qquad ...(5)$$

Solving equations (4) and (5), we get
$$x = 0, y = 0, x = 6 \text{ and } y = 4$$

Now, $$r = \frac{\partial^2 z}{\partial x^2} = 72xy^2 - 12x^2y^2 - 6xy^3 \qquad ...(6)$$
$$t = \frac{\partial^2 z}{\partial y^2} = 24x^3 - 2x^4 - 6x^3y \qquad ...(7)$$

Also $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(24x^3y - 2x^4y - 3x^3y^2)$$
$$s = 72x^2y - 8x^3y - 9x^2y^2 \qquad ...(8)$$

Thus at $x = 6$ and $y = 4$
$$r = 72 \times 6 \times 16 - 12 \times (36) \times (16) - 6 \times 6 \times 64$$
$$= -2054 = (-\text{ve})$$
$$t = 24(216) - 2(1216) - 6(216 \times 4) = -2592.$$

$\therefore$ $$s = 72(36)\,(4) - 8(216)\,(4) - 9(36) \times (16) = -1728$$

$\therefore$ $$rt - s^2 = (-2504) \times (-2592) - (-1728) > 0$$

Hence $rt - s^2 > 0, r < 0$

We have maximum at $x = 6, y = 4$.

Example 4.70. *Discuss the maxima and minima of the function, $f(x, y) = \sin x \sin y \sin(x + y)$.*

Solution. The points of maxima and minima are given by,

$$\frac{\partial f}{\partial x} = \sin y\,[\cos x \sin(x + y) + \sin x \cos(x + y)] = 0$$

$$\Rightarrow \quad \sin y\,[\sin(2x + y)] = 0 \qquad ...(1)$$

and

$$\frac{\partial f}{\partial y} = \sin x\,[\cos y \sin(x + y) + \sin y \cos(x + y) = 0$$

$$\Rightarrow \quad \sin x \sin(x + 2y) = 0 \qquad ...(2)$$

Solving these equations, the stationary points are (0,0) and $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = 2 \sin y \cos(2x + y),$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \sin y \cos(2x + y) + \cos y \sin(2x + y)$$

$$= \sin(2x + 2y),$$

and

$$t = \frac{\partial^2 f}{\partial y^2} = 2 \sin x \cos(x + 2y),$$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$$r = 2 \sin \frac{\pi}{3} \cos \pi = -\sqrt{3}$$

$$s = \sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$$

and

$$t = 2 \sin \frac{\pi}{3} \cos \pi = -\sqrt{3}.$$

Clearly $r < 0$ and $rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$. Hence $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and this maximum value is given by $f_{\max} = \sin \frac{\pi}{3} \sin \frac{\pi}{3} \sin \frac{2\pi}{3} = \frac{3\sqrt{3}}{8}$.

At (0, 0), we get $r = 1$, $s = t = 0$ such that $rt - s^2 = 0$. Thus no conclusion can be made at this stage and higher order terms must be considered.

However, at this point $f(x, y) = 0$, which may be minimum of the given function.

Example 4.71. *Discuss the maximum and minimum of $f(x, y) = x^2 + y^2 + 6x + 12$.*

Solution. Try yourself.

Answer. $f(x, y)$ is minimum when $x = -3$ and $y = 0$

Minimum value = 3

Example 4.72. *Discuss the maximum or minimum of the function $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$.*

Solution. Try yourself.

Answer: Function has a minimum at $x = y = a$. Also the minimum value $= 3a^2$.

Example 4.73. *Test the function $f(x, y) = (x^2 + y^2)\ e^{-(x^2+y^2)}$ for maximum and minimum for points not on the circle $x^2 + y^2 = 1$.*

Solution. Let $f(x, y) = (x^2 + y^2)\ e^{-(x^2+y^2)}$

$$\therefore \quad \frac{\partial f}{\partial x} = (x^2 + y^2)\ e^{-(x^2+y^2)} . (-2x) + 2x\, e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)}\ [1 - (x^2 + y^2)] . 2x$$

and
$$\frac{\partial f}{\partial y} = e^{-(x^2+y^2)}\ [1 - (x^2 + y^2)]2y$$

For maximum and minimum, we have

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \quad 2x[1 - (x^2 + y^2)]\, e^{-(x^2+y^2)} = 0$$

and
$$2y[1 - (x^2 + y^2)]\, e^{-(x^2+y^2)} = 0$$

$$\Rightarrow \quad x = 0, y = 0 \text{ and } x^2 + y^2 = 1$$

Leaving $x^2 + y^2 = 1$ as given, we take $x = 0, y = 0$.

Hence the point (0, 0) is only stationary point.

Again,
$$\frac{\partial^2 f}{\partial x^2} = r = (2 - 6x^2 - 2y^2)\ e^{-(x^2+y^2)} + (2x - 2x^3 - 2xy^2)\ e^{-(x^2+y^2)} . (-2x)$$

$$= e^{-(x^2+y^2)}\ [4x^4 - 10x^2 + 4x^2y^2 - 2y^2 + 2]$$

and
$$\frac{\partial^2 f}{\partial x \partial y} = s = (-4xy)\, e^{-(x^2+y^2)} + (2x - 2x^3 - 2xy^2)\ e^{-(x^2+y^2)} . (-2y)$$

$$= e^{-(x^2+y^2)}\ [-8xy + 4x^3y + 4xy^3]$$

and
$$\frac{\partial^2 f}{\partial y^2} = t = e^{-(x^2+y^2)}\ [2 - 2x^2 - 10y^2 - 4x^2y^2 + 4y^4]$$

At point (0, 0), $\quad r = 2, s = 0, t = 2$

$$rt - s^2 = 4 > 0, \text{ Also } r > 0.$$

$f(x, y)$ has a minimum value at (0, 0). Minimum value is $(0 + 0)e^{-0} = 0$

Example 4.74. *Find the maximum and minimum values of $x^4 + y^4 - x^2 + xy - y^2$.*

Solution. Let $f(x, y) = x^4 + y^4 - x^2 + xy - y^2$

For maximum or minimum,

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 2x + y = 0 \quad ...(1)$$

and
$$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + x - 2y = 0 \quad ...(2)$$

Subtracting (2) from (1), we have

$$4(x^3 - y^3) - 3(x - y) = 0$$

$$\Rightarrow \quad x - y = 0 \text{ or } 4(x^2 + y^2 + xy) - 3 = 0$$

Hence from equation (1), $x = y = 0$ or $\pm\,\dfrac{1}{2}$, Again by adding (1) and (2), we get

$$4(x^3 + y^3) - (x + y) = 0$$

$$\Rightarrow \qquad x + y = 0 \quad \text{or} \quad 4(x^2 + y^2 - xy) = 0$$

From equation (2), we get

$$x = -y = \pm\,\frac{\sqrt{3}}{2}$$

Further, we have, $r = \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 2$, $t = \dfrac{\partial^2 f}{\partial y^2} = 12y^2 - 2$ and $s = \dfrac{\partial^2 f}{\partial x \partial y} = 1$

Therefore

$$\begin{aligned} rt - s^2 &= 4(6x^2 - 1)(6y^2 - 1) - 1 \\ &= 3 > 0, \text{ when } x = 0, y = 0 \\ &= 0, \text{ when } x = y = \pm\,\frac{1}{2} \\ &= 48 > 0, \text{ when } x = -y = \pm\,\frac{\sqrt{3}}{2} \end{aligned}$$

Also

$$\begin{aligned} r &= -2 < 0, \text{ when } x = 0, y = 0 \\ &= 1 > 0, \text{ when } x = y = \pm\,\frac{1}{2} \\ &= 7 > 0, \text{ when } x = -y = \pm\,\frac{\sqrt{3}}{2}. \end{aligned}$$

Hence there is maximum, when $x = 0$, $y = 0$ and a minimum when $x = -y = \pm\,\dfrac{\sqrt{3}}{2}$, the values of $f(x, y)$, being zero and $-\dfrac{9}{8}$ respectively.

When $x = y = \pm\,\dfrac{1}{2}$, the test fails, in this case, proceed to higher degree terms of the Taylor's, expansion and get the result.

Hence there is a minimum when $x = y = \pm\,\dfrac{1}{2}$, the minimum value of '$f$' being $\left(-\dfrac{1}{8}\right)$.

4.9.1.2 Maxima and Minima of a Function of Three Independent Variables

Let $f(x, y, z)$ be a function of three independent variables and let $f(a, b, c)$ be an extreme value of the function at the point (a, b, c). Then by definition $f(a + h, b + k, c + l) - f(a, b, c)$ should preserve a fixed sign (positive for minimum and negative for a maximum) for all sufficiently small values of h, k and l.

Now from Taylor's expansion, we have $f(a + h, b + k, c + l) - f(a, b, c)$

$$= \frac{1}{2!}\,[A\,h^2 + B\,k^2 + C\,l^2 + 2\,Fkl + 2\,Glh + 2Hhk] + \text{higher order terms} \qquad ...(1)$$

When $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial z} = 0$ at $x = a$, $y = b$ and $z = c$,

where $A = \dfrac{\partial^2 f}{\partial x^2}$, $B = \dfrac{\partial^2 f}{\partial y^2}$, $C = \dfrac{\partial^2 f}{\partial z^2}$, $F = \dfrac{\partial^2 z}{\partial y \partial z}$, $G = \dfrac{\partial^2 f}{\partial z \partial x}$, $H = \dfrac{\partial^2 f}{\partial x \partial y}$ at the point (a, b, c).

Now by taking h, k and l sufficiently small the term within the square brackets in (1) can be made to govern the sign of the R.H.S. and therefore of L.H.S. If for all such values of h, k and l, this expansion is positive, then the function $f(x, y, z)$ has minimum at (a, b, c) and if the sign is negative then $f(x, y, z)$ has maximum at (a, b, c).

To identify the sign, let us write

$$Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk$$

$$= \frac{1}{A}[A^2h^2 + ABk^2 + ACl^2 + 2AFkl + 2AGlh + 2AHhk]$$

$$= \frac{1}{A}[(Ah + Hk + Gl)^2 + (AB - H^2)k^2 + 2(AF - GH)kl + (CA - G^2)l^2]$$

This will have the same sign as A, if $AB - H^2 > 0$, and $(AB - H^2)(CA - G^2) - (AF - GH)^2 > 0$ *i.e.*, if $AB - H^2 > 0$, and $A(ABC + 2FGH - AF^2 - BG^2 - CH^2) > 0$

We may also write these as

$$AB - H^2 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = D_1 \text{ (say)}$$

and
$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = D_2 \text{ (say)}$$

Hence the above expression will be positive if A, D_1, D_2 are all positive and will be negative, if these are alternately negative and positive. Hence the function $f(x, y, z)$ will have minimum, if $A > 0$, $D_1 > 0$ and $D_2 > 0$; and will have maximum, if $A < 0$, $D_1 > 0$, $D_2 < 0$.

Working Rule:

Let the function $u = f(x, y, z)$ have three independent variables x, y and z.

1. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, and equate to zero to obtain the values of x, y, z.

2. Find $A = \frac{\partial^2 u}{\partial x^2}$, $B = \frac{\partial^2 u}{\partial y^2}$, $C = \frac{\partial^2 u}{\partial z^2}$, $F = \frac{\partial^2 u}{\partial y \partial z}$, $G = \frac{\partial^2 u}{\partial z \partial x}$, $H = \frac{\partial^2 u}{\partial x \partial y}$

Now, we have the following cases:

a. The function $u = f(x, y, z)$ will be minimum at (a, b, c) if the expressions A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ be all positive at (a, b, c).

b. The function $u = f(x, y, z)$ will be maximum at (a, b, c) if the expressions A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ be alternately negative and positive.

c. If cases (a) and (b) are not satisfied, we have neither maximum nor minimum.

SOME SOLVED EXAMPLES

Example 4.75. *Show that the function $u = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ has minimum at $(1, 1, 1)$ and maximum at $(-1, -1, -1)$.*

Solution. We have $u = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ ...(1)

then, $\frac{\partial u}{\partial x} = 3(x + y + z)^2 - 3 - 24yz$ ...(2)

$\frac{\partial u}{\partial y} = 3(x + y + z)^2 - 3 - 24xz$ ...(3)

and $\frac{\partial u}{\partial z} = 3(x + y + z)^2 - 3 - 24xy$ ...(4)

for the maximum or minimum of u, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

From equation (2), (3) and (4), we get

$$x = y = z$$

Substituting $y = x$ and $z = x$ in (2), we get

$$27x^2 - 3 - 24x^2 = 0$$

$$\Rightarrow \quad x = \pm 1$$

$\Rightarrow x = y = z = 1$ and $x = y = z = -1$ are solution of (2), (3) and (4).

Hence the stationary points are (1, 1, 1) and (–1, –1, –1)

Now $A = \frac{\partial^2 u}{\partial x^2} = 6(x + y + z)$, $B = \frac{\partial^2 u}{\partial y^2} = 6(x + y + z)$

$C = \frac{\partial^2 u}{\partial z^2} = 6(x + y + z)$, $F = \frac{\partial^2 u}{\partial y \partial z} = 6(x + y + z) - 24x$

$G = \frac{\partial^2 u}{\partial z \partial x} = 6(x + y + z) - 24y$, $H = \frac{\partial^2 u}{\partial x \partial y} = 6(x + y + z) - 24z.$

For (1, 1, 1)

$A = 18, B = 18, C = 18, F = -6, G = -6$ and $H = -6$ at the point (1, 1, 1), we have

$$A = 18 > 0,$$

$$D_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 288 > 0$$

$$D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix} = 3426 > 0.$$

Since all three expressions are positive therefore we have u is minimum at the point (1, 1, 1).

Again for (–1, –1, –1), $A = -18, B = -18, C = -18, F = 6, G = 6$ and $H = 6$.

$\therefore$ At the point (–1, –1, –1) we have

$$A = -18 < 0,$$

$$D_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 288 > 0$$

and $$D_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix} = -3426 < 0$$

Hence, the above three expressions are alternately negative and positive.

Hence u is maximum at $(-1, -1, -1)$.

Example 4.76. *Prove that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.*

Solution. Try yourself.

Example 4.77. *Find the relative maximum/minimum value of the function $f(x, y) = 4x^2 + 9y^2 - 8x - 12y + 4$ within the rectangular region bounded by the lines $x = 2$, $y = 3$ and the co-ordinate axes.*

Solution. Try yourself.

Answers: Function has relative minimum value = – 12 at point $\left(1, \frac{2}{3}\right)$ in the given rectangular region.

Example 4.78. *The sum of the three positive numbers is constant. Show that their product is maximum when they are equal.*

Solution. Try yourself.

Example 4.79. *Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third, is maximum.*

Solution. Let 24 be divided in the parts x, y and z.

Then $$x + y + z = 24 \qquad ...(1)$$

Given that $$f(x, y, z) = x^3y^2z$$

$$\Rightarrow \quad f(x, y) = x^3y^2(24 - x - y) \qquad \text{[from (1)]}$$

or $$f(x, y) = 24x^3y^2 - x^4y^2 - x^3y^3$$

Now, $$\frac{\partial f}{\partial x} = 72x^2y^2 - 4x^3y^2 - 3x^2y^3,$$

$$\frac{\partial^2 f}{\partial x^2} = r = 144xy^2 - 12x^2y^2 - 6xy^3,$$

$$\frac{\partial f}{\partial y} = 48x^3y - 2x^4y - 3x^3y^2,$$

$$\frac{\partial^2 f}{\partial y^2} = t = 48x^3 - 2x^4 - 6x^3y \text{ and}$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 144x^2y - 8x^3y - 9x^2y^2$$

For maximum or minimum

$$\frac{\partial f}{\partial x} = 0$$

$$\Rightarrow \quad (72 - 4x - 3y)\, x^2y^2 = 0 \qquad ...(2)$$

and $$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad x^3y\,(48 - 2x - 3y) = 0$$

$$\Rightarrow \quad 48 - 2x - 3y = 0, x = 0, y = 0 \qquad ...(3)$$

Solving (2) and (3), we get $x = 12, y = 8$ etc.

At (12, 8), we have $r\,t - s^2 = +\text{ve}$ and $r < 0$, therefore $f(x, y)$ is maximum at (12, 8).

Put $x = 12$ and $y = 8$ in (1), we get $z = 4$. The value of $x = 12, y = 8$ and $z = 4$.

This is division of 24 for maximum $f(x, y, z)$

Example 4.80. *A rectangular box open at the top is to have given capacity, find the dimensions of the box requiring minimum material for its construction.*

Solution. Try yourself.

EXERCISE 4.7

1. Find the stationary points of the function $z = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and examine their nature.
2. Examine $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^3 + 72\,x$ for extreme values.
3. Find the maximum value of $f(x, y) = xye^{-(2x + 3y)}$.
4. Discuss the maximum and minimum of the following:

   i. $u(x, y) = x^2 + y^2 + \dfrac{2}{x} + \dfrac{2}{y}$

   ii. $u(x, y) = x^2 + xy + y^2 + 3x + 4y$

   iii. $f(x, y) = 2 \sin \dfrac{1}{2}(x + y) \cos \dfrac{1}{2}(x - y) + \cos (x + y)$

   iv. $f(x, y) = x^2 + y^2 + 6x + 12$.
5. Show that $\dfrac{a}{7}, \dfrac{2a}{7}, \dfrac{3a}{7}$ are stationary point of the function, $u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - 3xy^2z^4$ and determine whether, these are also an extreme point.
6. Find the maximum and minimum values of $u = y^2 + 2z^2 - 5x^4 + 4x^5$.
7. Find a plane triangle ABC such that $u = \sin^m A \sin^n B \sin^p C$ has maximum value.
8. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\dfrac{2r}{\sqrt{3}}$.
9. Find the absolute maximum and minimum values of $u(x, y) = 2 + 2x + 2y - x^2 - y^2$ on triangular plate in the first quadrant bounded by the lines $x = 0, y = 0$ and $y = 9 - x$.
10. Find the values of x and y, for which $x^2 + y^2 + 6x - 12 = 0$, has a minimum value and also find this minimum value.
11. A tree trunk of length l metres has the shape of a frustum of a circular cone with radii of its ends r_1 and r_2 metres, where $r_1 > r_2$. Find the length of a beam of uniform square cross-section which can be cut from the tree trunk so that the beam has the maximum volume.
12. The electric time constant of cylindrical coil of wire can be expressed approximately by $\alpha = \dfrac{mxyz}{ax + by + cz}$, where z is the axial length of the coil, y is the difference between the external and internal radii and x is the mean radius; a, b, c and m denote positive constants. If the volume of the coil is fixed, find the value of x and y which make the time constant, α as large as possible.

13. *ABCD* is a quadrilateral having no re-entrant angle and *P* is a point in its plane. Find the position of *P* for which the sum of the distances from the vertices is minimum.

14. In a plane ΔABC, find the maximum value of $\cos A \cos B \cos C$.

Hint. $A + B + C = \pi$

15. Examine the function $u(x, y) = x^3y^2(6 - x - y)$ for the maximum or minimum points not at origin.

16. Discuss the extremum values of the following:

i. $x^3 + y^3 + 3xy$

ii. $x^3 + y^3 - 63(x + y) + 12xy$

iii. $x^3y^2(1 - x - y)$

iv. $x^2 + 2xy + 2y^2 + 2x + y$

v. $\sin x + \sin y + \sin (x + y)$

vi. $x^3 - y^2 - 7x^2 + 4y + 15x - 13$

vii. $x^2y^2 - 5x^2 - 8xy - 5y^2$.

17. Discuss the maximum and minimum of $u = (ax^2 + by^2)^{-x^2 - y^2}$, $a \neq b$, what happens when $a = b = 1$?

18. Show the maximum value of $(ax + by + cz)\ e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}$ is $\left[\frac{1}{2e}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)\right]^{1/2}$.

Answers

1. $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$; z has minimum at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

2. (4, 0); 112 and (6, 0), 108.

3. $\left(\frac{1}{2}, \frac{1}{3}\right)$; $\frac{1}{6e^2}$; (0, 0), the function has maximum value at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

4. i. Minimum at (1, 1)

ii. Minimum at $\left(-\frac{2}{3}, -\frac{5}{3}\right)$

iii. Maximum at $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$

iv. At (–3, 0), minimum value = 3

5. maximum value = $\frac{108a^7}{7^7}$

6. minimum at (1, 0, 0); neither maximum nor minimum at (0, 0, 0).

7. u is maximum when = $\frac{\tan A}{m} = \frac{\tan B}{n} = \frac{\tan C}{p}$.

9. At (1, 1) maximum value = 4.

10. (–3, 0), minimum value = 3.

11. $\frac{8r_1^3 l}{27(r_1 - r_2)}$

13. *P* is the point of intersection of the diagonals of the quadrilateral.

14. At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, maximum value = $\frac{1}{8}$.

15. Maximum at (3, 2)

16. i. Maximum value = 1 at $(-1, -1)$.

ii. Maximum value = 784 at $(-7, -7)$

Minimum value = -216 at $(3, 3)$

iii. No extreme value at $(0, 0)$

Maximum value = $\dfrac{1}{432}$ at $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$

iv. Minimum value = $-\dfrac{5}{4}$ at $\left(-\dfrac{3}{2}, \dfrac{1}{2}\right)$

v. Maximum at $x = y = \dfrac{\pi}{3}$

vi. Neither maximum nor minimum at $(3, 2)$.

Maximum at $x = 5/3, y = 2$

vii. Maximum at $x = 0, y = 0$

Neither maximum nor minimum at $(3, 3)$, $(-3, -3)$ and $(1, -1)$.

17. Maximum or Minimum at $(0, 0)$ according as a and b both are negative or positive; Maximum or minimum at $(0, \pm 1)$ according as $b > 0$ and $a < b$ or $b < 0$ and $a > b$. Maximum or minimum at $(\pm 1, 0)$ according as $a > 0$ and $b < a$ or $a < 0$ and $b > a$. when $a = b = 1$, maximum at all points where $x^2 + y^2 = 1$ and minimum at $(0, 0)$.

Extreme Values-I

INTERESTING FACTS

- Maximum and minimum value that are found using Maxima and Minima are together known as Extrema (plural of extremum).
- A sculpture was displayed in the World Expo 2017 in Astana, Kazakhstan, which was named as Minima | Maxima, whose design was unique in itself.

REAL LIFE EXAMPLES (USES OF MAXIMA, MINIMA AND SADDLE POINTS)

- Applications in medicine in studying the effectiveness of drugs / spread of diseases (after how much time the maximum efficiency has been observed).
- Decay study in Nuclear energy sector.
- Population growth curve.
- In business, industry uses this concept by maximizing their profits or minimizing the loss, by estimating prices for items and also how many to keep in stock.
- A point of a function or surface which is a stationary point but not an extremum is called saddle point. E.g., a Handkerchief and monkey saddle are some of the surfaces that include saddle points.

4.9.2 Maximum and Minimum of Functions Subjected to Constraints

Here we shall consider the problems of determining the maxima and minima of functions of several variables in a different form.

Suppose
$$u = f(x_1, x_2, \ldots\ldots x_n) \qquad \ldots(1)$$
is a function of n variables $x_1, x_2, \ldots\ldots\ldots x_n$ which are not all independent, but are connected by means of m $(m < n)$ equations (constraints)

$$\left.\begin{array}{l}\phi_1(x_1, x_2, \ldots\ldots, x_n) = 0\\ \phi_2(x_1, x_2, \ldots\ldots, x_n) = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \phi_m(x_1, x_2, \ldots\ldots, x_n) = 0\end{array}\right\} \qquad \ldots(2)$$

So that only $n - m$ variables are independent and we can eliminate m dependent variables between (1) and m equations in (2). Thus u becomes a function of $(n - m)$ variables without any constraints and the problem is now equivalent to the evaluation of maxima and minima of a function of $(n - m)$ independent variables. The method of elimination of the variables is quite inconvenient and sometimes it is rather impossible. We, therefore, explain below the method known as Lagrange's method of undetermined multipliers.

4.9.3 Lagrange's Method of Undetermined Multipliers

For a maximum or minimum of u, we have

$$du = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \ldots\ldots + \frac{\partial f}{\partial x_n}\,dx_n = 0 \qquad \ldots(3)$$

where the differentials $dx_1, dx_2, \ldots\ldots, dx_n$ not all independent, but are connected by the equations,

$$\left.\begin{array}{l} d\phi_1 = \dfrac{\partial \phi_1}{\partial x_1}\,dx_1 + \dfrac{\partial \phi_1}{\partial x_2}\,dx_2 + \ldots\ldots + \dfrac{\partial \phi_1}{\partial x_n}\,dx_n = 0\\ d\phi_2 = \dfrac{\partial \phi_2}{\partial x_1}\,dx_1 + \dfrac{\partial \phi_2}{\partial x_2}\,dx_2 + \ldots\ldots + \dfrac{\partial \phi_2}{\partial x_n}\,dx_n = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ d\phi_m = \dfrac{\partial \phi_m}{\partial x_1}\,dx_1 + \dfrac{\partial \phi_m}{\partial x_2}\,dx_2 + \ldots\ldots + \dfrac{\partial \phi_m}{\partial x_n}\,dx_n = 0 \end{array}\right\} \qquad \ldots(4)$$

which is obtained by differentiating the m equations given in (2).

Multiplying equations (4) by some constraints $\lambda_1, \lambda_2, \ldots\ldots \lambda_m$ respectively and adding them to (3), we get

$$P_1\,dx_1 + P_2 dx_2 + \ldots\ldots\ldots + P_n dx_n = 0 \qquad \ldots(5)$$

where $$P_r = \frac{\partial f}{\partial x_r} + \lambda_1 \frac{\partial \phi_1}{\partial x_r} + \lambda_2 \frac{\partial \phi_2}{\partial x_r} + \ldots\ldots + \lambda_m \frac{\partial \phi_m}{\partial x_r}$$

Since $\lambda_1, \lambda_2, \ldots\ldots \lambda_m$ are arbitrary, these can be so chosen that they satisfy the m equations

$$P_1 = P_2 = \ldots\ldots = P_m = 0 \qquad \ldots(6)$$

Equation (5) then reduces to

$$P_{m+1}\,dx_{m+1} + P_{m+2}\,dx_{m+2} + \ldots\ldots + P_n\,dx_n = 0 \qquad \ldots(7)$$

As already explained, $(n - m)$ of the n variables are independent (no matter which of them are). we regard $x_{m+1}, x_{m+2}, \ldots\ldots x_n$ as independent variables, consequently the coefficients of $dx_{m+1}, dx_{m+2}, \ldots\ldots dx_n$ in equation (7) must be separately zero. Hence we must have

$$P_{m+1} = P_{m+2} = \ldots\ldots = P_n = 0$$

In this way, we obtain $(n + m)$ equations

$$P_1 = P_2 = \ldots\ldots = P_n = 0$$

and $\phi_1 = \phi_2 = \ldots\ldots = \phi_m = 0$

which determine the m multipliers $\lambda_1, \lambda_2, \ldots\ldots \lambda_m$ and the values of n variables $x_1, x_2, \ldots\ldots x_n$ for which the function $u = f(x_1, x_2, \ldots\ldots x_n)$ possesses a maximum or minimum under the constraints (2).

Remark: The above equations which determine the values of the variables for which the function u has a maximum or minimum may be conveniently written as follows:

Multiplying equations (4) by $\lambda_1, \lambda_2, \ldots\ldots \lambda_m$ respectively and add them to (3), we get

$$\left[\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial \phi_1}{\partial x_1} + \lambda_2 \frac{\partial \phi_2}{\partial x_1} + \ldots\ldots + \lambda_m \frac{\partial \phi_m}{\partial x_1}\right] dx_1$$

$$+ \left[\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial \phi_1}{\partial x_2} + \lambda_2 \frac{\partial \phi_2}{\partial x_2} + \ldots\ldots + \lambda_m \frac{\partial \phi_m}{\partial x_2}\right] dx_2 + \ldots\ldots$$

$$+ \left[\frac{\partial f}{\partial x_n} + \lambda_1 \frac{\partial \phi_1}{\partial x_n} + \lambda_2 \frac{\partial \phi_2}{\partial x_n} + \ldots\ldots + \lambda_m \frac{\partial \phi_m}{\partial x_n}\right] dx_n = 0$$

or $$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots\ldots + \frac{\partial f}{\partial x_n} dx_n = 0 \qquad \ldots(A)$$

where we have defined the function F by the relation

$$F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \ldots\ldots + \lambda_m \phi_m \qquad \ldots(B)$$

Evidently, F is the function of n variables $x_1, x_2, \ldots\ldots x_n$.

The constants $\lambda_1, \lambda_2, \ldots\ldots \lambda_m$, are at our choice, choose them to satisfy the m equations

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \ldots\ldots = \frac{\partial f}{\partial x_m} = 0 \qquad \ldots(C)$$

Equation (A) then reduces to

$$\frac{\partial f}{\partial x_{m+1}} dx_{m+1} + \frac{\partial f}{\partial x_{m+2}} dx_{m+2} + \ldots\ldots + \frac{\partial f}{\partial x_n} dx_n = 0 \qquad \ldots(D)$$

Since (n-m) of n variables are independent, let them be $x_{m+1}, x_{m+2}, \ldots\ldots, x_n$. Therefore, the coefficients of $dx_{m+1}, dx_{m+2}, \ldots\ldots, dx_n$ in (D) must be separately zero. Thus we must have

$$\frac{\partial f}{\partial x_{m+1}} = \frac{\partial f}{\partial x_{m+2}} = \ldots\ldots = \frac{\partial f}{\partial x_n} = 0.$$

Hence the conditions, that the function has a maximum or minimum are,

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \ldots\ldots = \frac{\partial f}{\partial x_n} = 0 \qquad \ldots(E)$$

which together with m equations (2) determine the required values of the variables.

But equations (E) are also the conditions that the function F regarded as a function of n independent variables $x_1, x_2, x_3, \ldots\ldots, x_n$ should have a maximum or minimum. It follows that the problem of maximizing or minimizing $u = f(x_1, x_2, \ldots\ldots, x_n)$ subject to m constraints is (2) is equivalent to finding of a maximum or minimum of F.

Working Rule: Let $u = F(x, y, z)$ be a function of three variables x, y, z which are connected by the relation $\phi(x, y, z) = 0$.

i. construct a new function $U = F(x, y, z) + \lambda\phi(x, y, z)$

ii. Obtain the equations $\dfrac{\partial U}{\partial x} = 0$, $\dfrac{\partial U}{\partial y} = 0$ and $\dfrac{\partial U}{\partial z} = 0$.

iii. Solve the above equations together with $\phi(x, y, z) = 0$. The values of x, y, z so obtained will give the stationary values of $f(x, y, z)$.

> **Remarks:** Although the lagrange's method is often very useful in applications yet the drawback is that we cannot determine the nature of the stationary point. This can sometimes be determined from physical consideration of the problems.

SOME SOLVED EXAMPLES

Example 4.81. *Discuss maximum and minimum of the function u (x, y, z) = sin x sin y sin z, where x, y, z are the angles of the triangle.*

Solution. Here we have,

$$u(x, y, z) = \sin x \sin y \sin z \qquad ...(1)$$

where the variables are connected by the relation $x + y + z = \pi$ ...(2)

For maximum or minimum of u, we have

$$du = \cos x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz = 0 \qquad ...(3)$$

Also from (2)

$$dx + dy + dz = 0 \qquad ...(4)$$

Multiplying (4) by λ and adding it to (3), then equating the coefficients of dx, dy, dz to zero, we get

$$\cos x \sin y \sin z + \lambda = 0$$
$$\sin x \cos y \sin z + \lambda = 0$$
$$\sin x \sin y \cos z + \lambda = 0$$

From these equations, we get

$$-\lambda = \cos x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$$

i.e., $\cot x = \cot y = \cot z$ (on dividing by $\sin x \sin y \sin z$)

i.e., $x = y = z = \dfrac{\pi}{3}$, by using (2)

Since only relation (2) is given among x, y, z therefore we may regard two variables say, x, y as independent and z as dependent. Now to determine a maximum or minimum, we have to find r, s, t, from (2), on differentiation, we get

$$1 + \frac{\partial z}{\partial x} = 0, \text{ and } 1 + \frac{\partial z}{\partial y} = 0$$

Now from (1),

$$\frac{\partial u}{\partial x} = \cos x \sin y \sin z + \sin x \sin y \cos z \frac{\partial z}{\partial x}$$

$$= \cos x \sin y \sin z - \sin x \sin y \cos z \qquad \left[\because \frac{\partial z}{\partial x} = -1\right]$$

$$= \sin y (\cos x \sin z - \sin x \cos z)$$

$$= \sin y \sin (z - x)$$

and $$\frac{\partial^2 u}{\partial x^2} = \sin y \cos (z-x)\left(\frac{\partial z}{\partial x}-1\right) = -2 \sin y \cos (z-x)$$

Similarly, $\dfrac{\partial^2 z}{\partial y^2} = -2 \sin x \cos (z-y)$

Now, $$\frac{\partial^2 z}{\partial x \partial y} = \cos y \sin (z-x) + \sin y \cos (z-x)\left(\frac{\partial z}{\partial y}\right)$$
$$= \cos y \sin (z-x) - \sin y \cos (z-x)$$
$$= \sin (z-x-y)$$

Hence putting $x = y = z = \dfrac{\pi}{3}$, we get

$$r = \frac{\partial^2 u}{\partial x^2} = -2 \sin \frac{\pi}{3} \cos 0 = -\sqrt{3}.$$

and $$t = -\sqrt{3}, \; s = \frac{\partial^2 u}{\partial y \partial x} = \sin\left(\frac{\pi}{3}-\frac{\pi}{3}-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$\therefore$ $$rt - s^2 = (-\sqrt{3})^2 - \left(\frac{-\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} > 0$$

Since $r < 0$ and $rt - s^2 > 0$, $\quad \therefore$ u is maximum at $x = y = z = \dfrac{\pi}{3}$.

Example 4.82. *Find the maximum or minimum value of $u = x^2 + y^2 + z^2$, subject to the condition $xy + yz + zx = 3a^2$.*

Solution. We have, $\quad u = x^2 + y^2 + z^2 \qquad ...(1)$

and $\quad \phi = xy + yz + zx - 3a^2 \qquad ...(2)$

Consider the lagrange's function,

$$F(x, y, z) = (x^2 + y^2 + z^2) + \lambda\,(xy + yz + zx - 3a^2)$$

For stationary values, $\quad dF = 0$

$$[2x + \lambda(y+z)]dx + [2y + \lambda(z+x)]\,dy + [2z + \lambda(x+y)]\,dz = 0$$

$\Rightarrow \quad 2x + \lambda(y+z) = 0 \qquad ...(3)$

$2y + \lambda(z+x) = 0 \qquad ...(4)$

$2z + \lambda(x+y) = 0 \qquad ...(5)$

Multiplying (3) by y, (4) by z and (5) by x, and adding, we get

$$2(xy + yz + zx) + \lambda\,(y^2 + yz + zx + z^2 + x^2 + yx) = 0$$

$\Rightarrow \quad 2(3a^2) + \lambda(u + 3a^2) = 0 \qquad ...(A)$

$[\because \quad xy + yz + zx = 3a^2$ and $u = x^2 + y^2 + z^2]$

Now, after multiplying (3) by x, (4) by y and (5) by z and adding, we get

$$u = -3a^2\lambda$$

Now (A)

$\Rightarrow \quad 6a^2 + \lambda(-3a^2\lambda + 3a^2) = 0$

$6a^2 - 3a^2\lambda\,(\lambda - 1) = 0$

$\Rightarrow \qquad 2 - \lambda^2 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1, 2$

Put $\lambda = -1$ in $\lambda = \dfrac{-u}{3a^2} \quad \Rightarrow \quad \dfrac{x^2+y^2+z^2}{3a^2} = 1$; $u = 3a^2$ is the maximum value

for $\lambda = 2$; $u = -6a^2$ is the minimum value.

Example 4.83. *Find the maximum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = P$.*

Solution. Let $\quad u(x, y, z) = x^2 + y^2 + z^2$,

and given that $ax + by + cz = P$ ...(1)

For maximum or minimum

$$du = 2xdx + 2ydy + 2zdz = 0$$

and $\quad adx + bdy + cdz = 0$

Therefore, using lagrange's method of undetermined multipliers,

$$2x + \lambda a = 0, 2y + \lambda b = 0, 2z + \lambda c = 0 \qquad ...(2)$$

Multiplying these equations by x, y, z respectively and adding, we get,

$$2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

$$\Rightarrow \qquad 2u_m + \lambda P = 0 \qquad ...(3)$$

where u_m indicates the minimum value of u. Also from (1) and (2), we get

$$a\left(-\frac{\lambda a}{2}\right) + b\left(-\frac{\lambda b}{2}\right) + c\left(-\frac{\lambda c}{2}\right) = P$$

$$\Rightarrow \qquad \lambda = \frac{-2P}{a^2+b^2+c^2}$$

Substituting these values in equation (3), we get

$$u_m = \frac{P^2}{(a^2+b^2+c^2)}$$

This is the minimum value (and not maximum) can be seen that from the geometrical consideration, u represents the square of the distance of a point on the plane (1) from the origin. This distance decreases as the point approaches from infinity towards the origin and after achieving a minimum again starts in creasing as the point moves away.

Example 4.84. *Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$.*

Solution. Here $\quad u = x^2 + y^2 + z^2$, ...(1)

and the relations between the variables x, y and z are given by

$$ax^2 + by^2 + cz^2 = 1 \qquad ...(2)$$

and $\quad lx + my + nz = 0$ ...(3)

For the maxima and minima, we have

$$du = 0$$

$$\Rightarrow \qquad 2x\,dx + 2ydy + 2zdz = 0$$

$$\Rightarrow \qquad xdx + ydy + zdz = 0 \qquad ...(4)$$

From (2) and (3), we get

$$axdx + bydy + czdz = 0 \qquad \text{...(5)}$$

and

$$ldx + mdy + ndz = 0 \qquad \text{...(6)}$$

Now multiplying (4) by 1, (5) by l_1 and (6) by l_2 and adding, we get

$$(xdx + ydy + zdz) + l_1(axdx + bydy + czdz) + l_2(ldx + mdy + ndz) = 0$$

or

$$(x + al_1x + ll_2)dx + (y + bl_1y + ml_2)dy + (z + c\,l_1z + nl_2)\,dz = 0$$

Now equating the coefficients of dx, dy, dz to zero, we get

$$x + l_1ax + l_2l = 0 \qquad \text{...(7)}$$

$$y + l_1by + l_2m = 0 \qquad \text{...(8)}$$

and

$$z + l_1cz + l_2n = 0 \qquad \text{...(9)}$$

multiplying the equations (7), (8) and (9) by x, y and z respectively, then adding, we get

$$(x^2 + y^2 + z^2) + l_1(ax^2 + by^2 + cz^2) + l_2(lx + my + nz) = 0$$

$\Rightarrow$ $\quad u + l_1.1 + l_2.(0) = 0$ (from (1), (2) and (3))

$$\Rightarrow \quad u = -l_1$$

Putting for l_1 in the equations (7), (8) and (9) we get

$$x = \frac{l_2l}{au-1},\ y = \frac{l_2m}{bu-1},\ z = \frac{l_2n}{cu-1} \qquad \text{...(10)}$$

From (3) and (10), we get

$$\frac{l_2l^2}{au-1} + \frac{l_2m^2}{bu-1} + \frac{l_2n^2}{cu-1} = 0$$

$$\Rightarrow \quad \frac{l^2}{au-1} + \frac{m^2}{bu-1} + \frac{n^2}{cu-1} = 0 \qquad \text{...(11)}$$

which gives the maximum and minimum of

$$u = x^2 + y^2 + z^2$$

> **Note:** (*i*) The equation (11) is a quadratic in u. So it gives two stationary values of u.
>
> (*ii*) Geometrically, the surface $ax^2 + by^2 + cz^2 = 1$ denotes an ellipsoid whose center is origin and $lx + my + nz = 0$ denotes a plane passing through the origin. The points (x, y, z) satisfying both the conditions (2) and (3) lie on the conic in which (2) and (3) intersect. The maximum value of this distance is the major axis of this conic and the minimum value of this distance is the minor axis of this conic. Hence, equation (11) gives the squares of the lengths of the semi-axes of the conic of intersection.

Example 4.85. *Find the maximum and minimum value of* $u = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}$ *subject to the conditions* $lx + my + nz = 0$ *and* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Solution. Let us define $F = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4} + \lambda_1(lx + my + nz) + \lambda_2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right)$ for maxima or minima of F, we have

$$\frac{\partial F}{\partial x} = \frac{2x}{a^4} + \lambda_1 . l + \lambda_2\frac{2x}{a^2} = 0 \qquad \text{...(1)}$$

$$\frac{\partial F}{\partial y} = \frac{2y}{b^4} + \lambda_1 . m + \lambda_2 \frac{2y}{b^2} = 0 \qquad ...(2)$$

and
$$\frac{\partial F}{\partial z} = \frac{2z}{c^4} + \lambda_1 . n + \lambda_2 \frac{2z}{c^2} = 0 \qquad ...(3)$$

Multiplying (1) by x, (2) by y and (3) by z, and adding, we get

$$2u + \lambda_1(0) + 2\lambda_2 = 0 \qquad \Rightarrow \quad \lambda_2 = -u.$$

$\therefore$ From (1),

$$\frac{2x}{a^4} + \lambda_1 . l - \frac{2ux}{a^2} = 0$$

$\Rightarrow$
$$x = -\frac{\lambda_1 \, l \, a^4}{2(1 - ua^2)}$$

Similarly,
$$y = -\frac{\lambda_1 \, m \, b^4}{2(1 - ub^2)}$$

and
$$z = -\frac{\lambda_1 \, n \, c^4}{2(1 - uc^2)}$$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$\frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} = 0 \qquad ...(4)$$

This equation gives the required maximum or minimum value of u.

Geometrical Interpretation; $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ is the reciprocal of the square of the distance of the tangent plane $\frac{Xx}{a^2} + \frac{Yy}{b^2} + \frac{Zz}{c^2} = 1$ to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ from the origin. If the point (x, y, z) lies on the plane $lx + my + nz = 0$, then the problem is to find the maximum or minimum of the reciprocal of the square of the distance of the tangent plane to the ellipsoid from the origin at the points common to the plane $lx + my + nz = 0$ and the ellipsoid.

Example 4.86. *Using the lagrange's method (of undetermined multipliers). Find shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.*

Solution. Let a point (x, y, z) be on the sphere, then its distance from (1, 2, 2) is

$$F = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Obviously, the distance and the square of the distance have extremum values for the same values of x, y, z.

Consider $\qquad F(x, y, z) = \{(x-1)^2 + (y-2)^2 + (z-2)^2\} + \lambda(x^2 + y^2 + z^2 - 36)$

where λ is a multiplier. For the stationary value of F,

$$\frac{\partial F}{\partial x} = 0, \ \frac{\partial F}{\partial y} = 0, \ \frac{\partial F}{\partial z} = 0$$

$\therefore$
$$\frac{\partial F}{\partial x} = 0, \ 2(x-1) + 2\lambda x = 0$$

$$\frac{\partial F}{\partial y} = 0,\ 2(y-2) + 2\lambda y = 0$$

and $$\frac{\partial F}{\partial z} = 0,\ 2(z-2) + 2\lambda z = 0$$

From these, we obtain, $x = \frac{1}{1+\lambda}$, $y = \frac{2}{1+\lambda}$, $z = \frac{2}{1+\lambda}$ putting these values in equation $x^2 + y^2 + z^2 = 36$, we obtain

$$\frac{1}{(1+\lambda)^2} + \frac{4}{(1+\lambda)^2} + \frac{4}{(1+\lambda)^2} = 36.$$

$$\Rightarrow \qquad (1 + \lambda) = \pm\frac{1}{2} \text{ giving } \lambda = -\frac{1}{2}, -\frac{3}{2}.$$

Hence, the points are (2, 4, 4) and (–2, –4, –4), thus the minimum distance

$$= \sqrt{(1-2)^2 + (2-4)^2 + (2-4)^2}$$

$$= \sqrt{(-1)^2 + (-2)^2 + (-2)^2}$$

$$= \sqrt{9} = 3.$$

and the maximum value is

$$= \sqrt{(1+2)^2 + (2+4)^2 + (2+4)^2}$$

$$= \sqrt{9+36+36} = \sqrt{81} = 9.$$

Example 4.87. *If two variables x and y are connected by the relation $ax^2 + by^2 = ab$, show that the maximum and minimum values of the function $x^2 + y^2 + xy$ will be the values of u given by the relation $4(u - a)(u - b) = ab$.*

Solution. Try yourself.

Example 4.88. *Find the maximum or minimum values of $x^p y^q z^r$ subject to the condition*

$$ax + by + cz = p + q + r.$$

Solution. Try yourself.

Example 4.89. *Let t be the temperature at any point (x, y, z) in space $s = 400xyz^2$. Find the maximum temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.*

Solution. Let $s = 400\,xyz^2$ and $x^2 + y^2 + z^2 = 1$

Let $$F(x, y, z) = x^2 + y^2 + z^2 - 1$$

Consider the lagrange's method, we have

$$\frac{\partial s}{\partial x} + \lambda\frac{\partial F}{\partial x} = 0$$

$$\Rightarrow \qquad 400yz^2 + \lambda(2x) = 0 \qquad ...(1)$$

$$\frac{\partial s}{\partial y} + \lambda\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \qquad 400xz^2 + (2y)\lambda = 0 \qquad ...(2)$$

and $$\frac{\partial s}{\partial z} + \lambda\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow \qquad 800xyz + \lambda(2z) = 0 \qquad ...(3)$$

Multiplying equation (1) by x, (2)by y and (3) by z and adding, we have

$$1600xyz^2 + 2\lambda(x^2 + y^2 + z^2) = 0$$

$$\Rightarrow \qquad 1600\, xyz^2 + 2\lambda(1) = 0$$

$$\Rightarrow \qquad \lambda = -800\, xyz^2$$

substituting the value of λ in (1), we get

$$400\, yz^2 + 2x\,(-\,800\, xy\, z^2) = 0$$

$$1 - 4x^2 = 0 \qquad \Rightarrow \quad x = \pm\,\frac{1}{2}$$

Similarly $\qquad y = \pm\,\frac{1}{2}$

Substituting the value of λ in equation (3), we get

$$800\, xyz - 1600\, xy\, z^3 = 0$$

$$1 - 2z^2 = 0 \qquad \Rightarrow \quad z = \pm\,\frac{1}{\sqrt{2}}.$$

Substituting the value of x, y and z in s, we get

$$s = 400 \times \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = 50.$$

Example 4.90. *Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Solution. Try yourself.

Example 4.91. *Obtain the extreme values of* $x^3 + y^3 + z^3 + 3kxyz$, $k \neq 2$, *when* x, y, z *are subject to condition* $x + y + z = 1$ *and prove that the symmetrical extreme value is maximum or minimum according as* $k > 2$ *or* $k < 2$.

Solution. We have $\qquad u = x^3 + y^3 + z^3 + 3kxyz,$

where $\qquad x + y + z = 1$

Consider the Lagrange's function,

$$f(x, y, z) = x^3 + y^3 + z^3 + 3kxyz + \lambda(x + y + z - 1)$$

For extreme values of u, $\; df = 0$

$$\Rightarrow \qquad \frac{\partial f}{\partial x} = 3(x^2 + kyz) + \lambda = 0 \qquad ...(1)$$

$$\frac{\partial f}{\partial y} = 3(y^2 + kzx) + \lambda = 0 \qquad ...(2)$$

and $$\frac{\partial f}{\partial z} = 3(z^2 + kxy) + \lambda = 0 \qquad ...(3)$$

Subtracting (2) from (1), we get

$$(x - y)\,(x + y - kz) = 0$$

and subtracting (3) from (1), we get

$$(x - z)\,(x + z - ky) = 0$$

Therefore either i. $x = y = z = \frac{1}{3}$ or

ii. $x = y = \frac{z}{k-1} = \frac{1}{k+1}$

iii. $x = z = \frac{y}{k-1} = \frac{1}{k+1}$ or

iv. $y = z = \frac{x}{k-1} = \frac{1}{k+1}$, The solution (i) is symmetrical in x, y, z giving $u = \frac{1}{9}(1 + k)$; the solutions (ii). (iii) and (iv) are non symmetrical but make $u = \frac{k^3+1}{(1+k)^3}$ in each case. It will be noticed that if $k = 2$, there is only one solution, viz. (i).

Further, we have, $d^2f = 6(xdx^2 + ydy^2) + 6k(x\,dydz + ydzdx + zdxdy)$

$= 2(dx^2 + dy^2 + dz^2) + 2x(dydz + dzdx + dxdy)$ (from (1))

But $dx + dy + dz + 0$, for all $x + y + z = 1$

Therefore $dx^2 + dy^2 + dz^2 = -2\,(dydz + dzdx + dxdy)$

Now, it is clear that d^2f is negative when $k > 2$ and $+\,ve$ when $k < 2$.

Hence the value of u in the symmetrical case is maximum or minimum according as $k > 2$ or $k < 2$.

Example 4.92. *Find the stationary points of $f(x, y) = y^2 + 4xy + 3x^2 + x^3$. Examine them for the extreme values of the function.*

Solution. Try yourself.

Answer: Minimum value at point $\left(\frac{2}{3}, \frac{-4}{3}\right)$ minimum value is $\frac{-4}{37}$.

Example 4.93 *Find the minimum value of $x^2 + y^2$ subject to the condition $ax + by - c = 0$.*

Hint: Consider Lagrange's function

$$f(x, y) = x^2 + y^2 + \lambda(ax + by - c)$$

Now $\frac{\partial f}{\partial x} = 2x + \lambda a = 0$ and $\frac{\partial f}{\partial y} = 2y + \lambda b = 0$

Solving $x = -\frac{1}{2}a\lambda,\ y = \frac{-1}{2}b\lambda.$

Putting these values in $ax + by = c$, we find $\lambda = \frac{-2c}{a^2+b^2}$.

It gives the minimum value of $(x^2 + y^2)$ at $\left(-\frac{a}{2}\lambda, \frac{-b}{2}\lambda\right)$, where $\lambda = \frac{-2c}{a^2+b^2}$.

Example 4.94. *Find the minimum value of the function $f(x, y) = x^2 + y^2 + z^2 + xy + ax + by$, where a, b are constant.*

Hint: $\frac{\partial f}{\partial x} = 2x + y + a = 0$ (for maximum or minimum)

and $\frac{\partial f}{\partial y} = 2y + x + b = 0$

solving these equations, we get

$$x = \frac{b-2a}{3}, y = \frac{a-2b}{3}.$$

Again, $$r = \frac{\partial^2 f}{\partial x^2} = 2, s = \frac{\partial^2 f}{\partial x \partial y} = 1, t = \frac{\partial^2 f}{\partial y^2} = 2$$

Hence at the point $\left(\frac{b-2a}{3}, \frac{a-2b}{3}\right)$ the given function has a minimum and the minimum value is $\frac{-(3a^2+3b^2-2ab)}{9}$.

Example 4.95. *Find the maximum or minimum value of u, where* $u = \frac{xyz}{(x+a)(x+y)(y+z)(z+b)}$.

Hint: Taking log on both sides, we have

$$\log u = \log x + \log y + \log z - \log(x+a) - \log(x+y)$$
$$- \log(y+z) - \log(z+b)$$

Now, on differentiating $\frac{1}{u}\frac{\partial u}{\partial x} = \frac{ay-x^2}{x(x+a)(x+y)}$

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{xz-y^2}{y(x+y)(y+z)}$$

and $$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{by-z^2}{z(y+z)(z+b)}$$

For maximum or minimum,

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ $x^2 = ay, y^2 = zx, z^2 = \text{by}$

$\Rightarrow$ a, x, y; x, y, z and y, z, b are all in geometrical progression

$\therefore$ a, x, y, z, b are in geometrical progression

Let r be the common ratio.

Then, $b = ar^4, x = a\,r, y = ar^2, z = ar^3,$

To find A, B, C, D, F, G, H, we put

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = -\text{ve}, A < 0, AB - H^2 > 0.$$

Hence u is maximum and maximum value of $u = \frac{1}{(a^{1/4}+b^{1/4})^4}$.

Example 4.96. *If* $u = ax^2 + by^2 + cz^2$, *where* $x^2 + y^2 + z^2 = 1$ *and* $lx + my + nz = 0$, *show that stationary values of u satisfy the equation* $\frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$

Hint: Let $u = ax^2 + by^2 + cz^2, f = x^2 + y^2 + z^2 - 1, g = lx + my + nz$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2ax,\ \frac{\partial u}{\partial y} = 2by,\ \frac{\partial u}{\partial z} = 2zc$$

$$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\ \frac{\partial f}{\partial z} = 2z$$

and $$\frac{\partial g}{\partial x} = l,\ \frac{\partial g}{\partial y} = m,\ \frac{\partial g}{\partial z} = n$$

Then by lagrange's method of multiplier, we have

$$\frac{\partial u}{\partial x} + \lambda_1 \frac{\partial f}{\partial x} + \lambda_2 \frac{\partial g}{\partial x} = 0$$

$$\Rightarrow \quad 2ax + 2x\lambda_1 + \lambda_2 l = 0 \qquad ...(1)$$

Similarly $2by + 2y\lambda_1 + \lambda_2 m = 0$ ...(2)

and $2cz + 2z\lambda_1 + \lambda_2 n = 0$ ...(3)

Multiply (1), (2), (3) by x, y, and z respectively and adding, we get

$$\lambda_1 = -u$$

Putting the value of λ_1 in (1), (2) and (3), we get

$$x = -\frac{\lambda_2 l}{2(a-u)},\ y = -\frac{\lambda_2 m}{2(b-u)} \text{ and } z = -\frac{\lambda_2 n}{2(c-u)}$$

Putting these values of x, y, z in $lx + my + nz = 0$, we get

$$-\frac{\lambda_2 l^2}{2(a-u)} - \frac{\lambda_2 m^2}{2(b-u)} - \frac{\lambda_2 n^2}{2(c-u)} = 0$$

$$\Rightarrow \quad \frac{l^2}{a-u} + \frac{m^2}{b-u} + \frac{n^2}{c-u} = 0.$$

Example 4.97. *Find the point on the surface $z^2 = xy + 1$ nearest to origin.*

Hint: Let (x, y, z) be a point from the origin, $z^2 = xy + 1$. ...(1)

Let s be the distance of this point from the origin, $s = \sqrt{x^2 + y^2 + z^2}$.

$$\Rightarrow \quad f(x, y, z) = s^2 = x^2 + y^2 + z^2,$$

then $$f(x, y, z) = x^2 + y^2 + xy + 1, \text{ from (1)}$$

Now, $$\frac{\partial f}{\partial x} = 2x + y,\ r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial f}{\partial y} = 2y + x,\ t = \frac{\partial^2 f}{\partial y^2} = 2$$

and $$s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

For maximum or minimum

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0 \qquad ...(2)$$

$$\Rightarrow \quad 2x + y = 0 \text{ and } 2y + x = 0. \qquad ...(3)$$

Solving (2) and (3), we get $x = 0, y = 0$.

Here $rt - s^2 = 3$ (+ ve) and also r is positive.

$\therefore f(x, y)$ is minimum at (0, 0). Put $x = 0, y = 0$ in (1), we get $z = \pm 1$.

Hence the points on $z^2 - xy - 1 = 0$ nearest to origin are $(0, 0, \pm 1)$.

Example 4.98. *Find the maximum and minimum of $x^2 + y^2$, subject to the condition $ax^2 + by^2 + 2hxy = 1$.*

Hint: We have $\quad u = x^2 + y^2$ where x, y connected by the relation are ...(1)

$$ax^2 + by^2 + 2hxy = 1 \qquad \text{...(2)}$$

For maximum or minimum of u, $du = 0$

$$\Rightarrow \quad 2xdx + 2ydy = 0 \quad \Rightarrow \quad x\,dx + y\,dx = 0 \qquad \text{...(3)}$$

and $\quad (ax + hy)\,dx + (hx + by)\,dy = 0$ ...(4)

Then by using lagrange's method of multipliers, we have

$$x + \lambda(ax + hy) = 0 \qquad \text{...(5)}$$

and $\quad y + \lambda(hx + by) = 0$ ...(6)

Multiplying (5) by x and (6) by y and adding, we get

$$x^2 + y^2 + \{\lambda\,(ax^2 + 2hxy + by^2)\} = 0$$

$$\Rightarrow \quad u + \lambda = 0 \quad \Rightarrow \quad \lambda = -u$$

From (5), $\quad x - u\,(ax + hy) = 0$

$$\Rightarrow \quad x\,(1 - au) - huy = 0 \quad \Rightarrow \quad \left(a - \frac{1}{u}\right) x + hy = 0 \qquad \text{...(7)}$$

and $\quad hx + \left(b - \frac{1}{u}\right) y = 0$ ...(8)

Eliminating x and y from (7) and (8), we get

$$\begin{vmatrix} a - \frac{1}{u} & h \\ h & b - \frac{1}{u} \end{vmatrix} = 0 \quad \Rightarrow \quad \left(a - \frac{1}{u}\right)\left(b - \frac{1}{u}\right) = h^2 \qquad \text{...(9)}$$

Hence the required maximum or minimum values of $u = x^2 + y^2$ are the roots of (9).

EXERCISE 4.8

1. Find the minimum value of $x + y + z$, subject to the condition $\left(\frac{a}{x}\right) + \left(\frac{b}{y}\right) + \left(\frac{c}{z}\right) = 1$.
2. Given $F = \dfrac{5xyz}{x + 2y + 4z}$. Find the values of x, y, z for which F is maximum subject to the condition $xyz = 8$.
3. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = d$.
4. Find the stationary value of $u = x^2 + y^2 + z^2$ subject to the condition $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$.

5. Using the method of multipliers, find the maximum product of the numbers, x, y and z when $x^2 + y^2 + z^2 = 16$.

6. Find the minimum and maximum distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.

7. Discuss the maximum and minimum of $x^2 + y^2 + z^2$ subject to the following conditions

i. $ax^2 + by^2 + cz^2 = 1$ ii. $yz + zx + xy = 3a^2$

iii. $x + y + z = 3a$ iv. $xyz = a^3$

8. Show that if the perimeter of a triangle is constant, its area is maximum when it is equilateral.

9. Use the lagrange's method of multipliers to find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1$, $xyz = -1$.

10. Find the shortest and largest distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.

11. Find the maximum and minimum values of $x^2 + y^2 + z^2$, subject to condition $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$.

12. If $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots\ldots + \frac{x_n^2}{a_n^2} = 1$, where $a_1^2 > a_2^2 > \ldots\ldots > a_n^2$, Prove that the stationary value of $x_1^2 + x_2^2 + \ldots\ldots + x_n^2$ are $a_1^2\ a_2^2 \ldots\ldots a_n^2$, out of these value, prove that only a_1^2 is true maximum and a_n^2 is true minimum.

13. Find the maximum value of $x^a y^b z^c$ subject to $x + y + z = \lambda$, $x > 0$, $y > 0$, $z > 0$ where a, b, c are positive constants.

14. Find the triangular pyramid of given base and altitude which has the least surface.

Answers

1. $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$

2. Maximum at $x = 4$, $y = 2$, $z = 1$

3. Minimum value at $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$; $\frac{d}{a^2 + b^2 + c^2}$

4. Stationary values are $\frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0$

5. $3\sqrt{3}$

6. Minimum distance = $\sqrt{6}$ and maximum distance = $3\sqrt{6}$.

7. i. Maximum and minimum are the roots of $\left(\frac{1}{a} - u\right)\left(\frac{1}{b} - u\right)\left(\frac{1}{c} - u\right) = 0$

ii. Minimum value = $3a^3$ at (a, a, a)

iii. Minimum value = $3a^3$ at (a, a, a)

iv. Minimum value = $3a^3$ at (a, a, a)

9. Minimum value = 3

10. Shortest distance = 1, greatest distance = 4

11. Maximum or minimum value of u are given by $\begin{vmatrix} a - u^{-1} & h & g \\ h & b - u^{-1} & f \\ g & f & c - u^{-1} \end{vmatrix} = 0$

12. Stationary points are given by $\begin{vmatrix} 0 & c & b & \dfrac{2u}{n} \\ c & 0 & a & \dfrac{2u}{n} \\ b & a & 0 & \dfrac{2u}{n} \\ 1 & 1 & 1 & n \end{vmatrix}$, value is maximum

Lagrange Multipliers

13. $a^a\, b^b\, c^c \left(\dfrac{\lambda}{a+b+c} \right)^{a+b+c}$

14. The foot of the altitude is the incentre of the base and the minimum surface (excluding the base) of the triangular pyramid is $(p^2 s^2 + \Delta^2)^{1/2}$, where p is the altitude, s is the semi-perimeter of the base and Δ is area.

INTERESTING FACTS

- In economics, the Lagrange multiplier λ has a specific meaning. If you're maximizing profit subject to a limited resource, λ is that resource's marginal value often referred as the "shadow price" of the resource.
- Many designers at cold drink companies use this concept to minimize the aluminium used while making sure that it contains specified amount of drink.

REAL LIFE EXAMPLES

- It is used to find the local maxima or minima of a function.
- In economics, this concept is used to increase the production at the point of maximization with respect to the increase in the value of the inputs.
- This technique is used to solve some optimization problems: those having one or several equality constraints.
- This technique is also used in SVM (Support Vector Machines).
- It is also used to solve some extreme value problems in science, economics,and engineering.
- Many computational programming methods, such as the barrier and interior point method, penalizing and augmented Lagrange method, etc, they have been developed on the basic rules of Lagrange multipliers method.

4.10 VECTORS

Introduction

This topic deals with vectors and vector functions in 3D-space and extends the differential calculus to these vector function forces, velocities and various other quantities are vectors. This makes the algebra and calculus of these vector functions the natural instrument for the engineer and physicist in solid mechanics, fluid flow, heat flow, electrostatics, and so on. The engineer must understand these fields as the basis of the design and construction of system or robots. In three dimensions (as opposed to higher dimensions), geometrical ideas become influential, enriching the theory and many geometrical quantities (tangents and normals, for example) can be given by vectors.

We first explain the basic algebraic operations with vectors in 3D-space. Vector differential calculus begins next with a discussion of vector functions, which represent vector fields and have various physical and geometrical applications. Then the basic concepts of differential calculus are extended to vector functions in a simple and natural fashion. Vector functions are useful in studying curves and their applications as paths or moving bodies in mechanics.

We finally discuss three physically and geometrically important concepts related to scalar and vector fields namely the gradient, divergence and curl.

4.10.1 Scalar And Vector Quantities

A quantity which has magnitude as well as direction is called vector and a quantity which has only magnitude and no direction is called scalar. A vector is represented by a letter with an arrow over it such as $\vec{a}$ and its magnitude by $|\vec{a}|$. In this topic, we give working knowledge of vectors. Some of these have wide application in the field of science and Engineering.

4.10.2 Scalar and Vector Product, Angular velocity

a. **Scalar Product:** If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then the scalar product of these vectors is defined as $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

b. **Vector Product or Cross Product:** If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then the magnitude of cross product of $\vec{a}$ and $\vec{b}$ is defined as $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$ and its direction is perpendicular to both $\vec{a}$ and $\vec{b}$ and forms a right handed system.

It $\hat{n}$ be a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\ \sin\theta\,\hat{n}$$

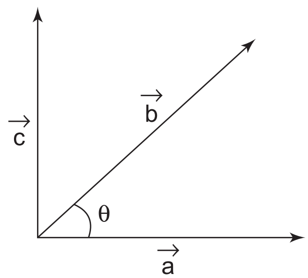


Fig. 4.1

c. **Angular Velocity:** Let a rigid body be rotating about the axis OA with the angular velocity $\vec{\omega}$ which is a vector.
Let P be any point on the body such that,

$$\vec{OP} = \vec{r}\ , AOP = \theta \quad \text{and} \quad AP \perp OA$$

Let V be the velocity of P and $\hat{n}$ be a unit vector perpendicular to $\vec{\omega}$ and $\vec{r}$

then $$\vec{\omega}\times\vec{r} = (\omega r\sin\theta)\ \hat{n} = (\omega AP)\,\hat{n}$$

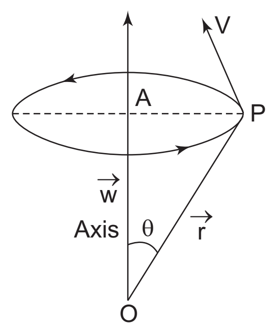


Fig. 4.2

$$= (\text{speed of } P)\,\hat{n} = \text{velocity of } P \perp \text{ to } \vec{\omega} \text{ and } \vec{r}$$

$$\therefore \qquad \vec{v} = \vec{\omega} \times \vec{r}$$

4.10.3 Differentiation of Vectors

Let O be the origin and P be the position of a moving particle at time 't'.

Let $\qquad \vec{OP} = \vec{r}$

Let Q be the position of the particle at time $t + \delta t$ such that

$$\vec{OQ} = \vec{r} + \vec{\delta r}$$

$$\therefore \qquad \vec{PQ} = \vec{OQ} - \vec{OP} = (\vec{r} + \delta\vec{r}) - \vec{r} = \delta\vec{r}$$

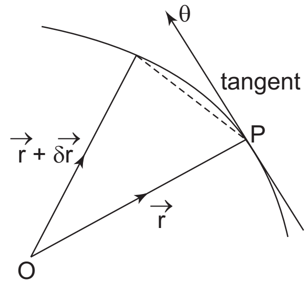


Fig. 4.3

$\dfrac{\delta\vec{r}}{\delta t}$ is a vector. As $\delta t \to 0$, Q tends to P and the chord PQ becomes tangent at P.

We define $\dfrac{d\vec{r}}{dt} = \lim\limits_{\delta t \to 0} \dfrac{\delta\vec{r}}{dt}$, then $\dfrac{d\vec{r}}{dt}$ is the vector in the direction of the tangent at P. $\dfrac{d\vec{r}}{dt}$ is also called the differential coefficients of $\vec{r}$ w.r.t. 't'.

4.10.4 Formulae of Differentiation

i. $\quad \dfrac{d}{dt}(\vec{F} + \vec{G}) = \dfrac{d\vec{F}}{dt} + \dfrac{d\vec{G}}{dt}$

ii. $\quad \dfrac{d}{dt}(\vec{F}\,\phi) = \dfrac{d\vec{F}}{dt}\phi + \vec{F}\dfrac{d\phi}{dt}$

iii. $\quad \dfrac{d}{dt}(\vec{F}\,.\,\vec{G}) = \vec{F}\,.\,\dfrac{d\vec{G}}{dt} + \dfrac{d\vec{F}}{dt}\,.\,\vec{G}$

iv. $\quad \dfrac{d}{dt}(\vec{F} \times \vec{G}) = \vec{F} \times \dfrac{d\vec{G}}{dt} + \dfrac{d\vec{F}}{dt} \times \vec{G}$

4.10.5 Scalar and Vector Point Functions

i. Field: If a function is defined in any region of space for every point of the region, then this region is known as field.

ii. Scalar Point Function: A function $\phi(x, y, z)$ is called a scalar point function if it associates a scalar with every point in space *e.g.* temperature distribution in a heated body, density of a body, speed of a body are some examples of a scalar point functions.

iii. **Vector Point Function:** If a function $F(x, y, z)$ defines a vector at every point in a region, then $F(x, y, z)$ is called a vector point function. The velocity of a moving fluid, gravitational forces are examples of a vector point function.

4.11 VECTOR DIFFERENTIAL OPERATOR DEL *i.e.*, ∇

The operator ∇ is defined as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

4.11.1 Gradient

The gradient of a scalar point function ϕ is denoted by $\nabla\phi$, and is written as gradient ϕ and is defined as,

$$\text{grad } \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

This shows that gradient ϕ is a vector quantity.

Geometrical Interpretation of Gradient

We know, $\nabla \phi = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$

Let a level surface (the surface on which the value of the function remains the same) $\phi(x, y, z) = k$ passes through a point 'P' whose position vector is $\vec{R}$.

Let another level surface $\phi + \delta\phi = k'$ be a neighbouring surface which passes through Q whose position vector is $\vec{R} + \delta\vec{R}$

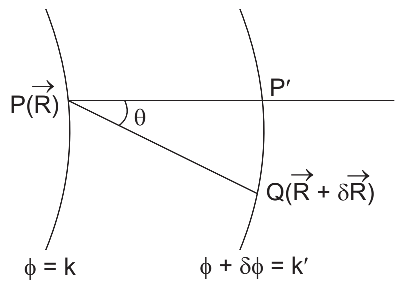


Fig. 4.4

$\therefore$ $\quad \vec{PQ} = \delta\vec{R}$

Now, we have, $\nabla\phi \,.\, \delta\vec{R} = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right).(\delta x\,\hat{i} + \delta y\,\hat{j} + \delta\hat{z}\,\hat{k}) \quad [\because \quad \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}\,]$

$$= \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = \delta\phi$$

$$\nabla\phi \,.\, \delta\vec{R} = \delta\phi$$

Now, there are two cases:

Case I: When Q lies on the surface on which P lies, then $\delta\phi = 0$, $\Rightarrow$ $\nabla\phi \,.\, \delta\vec{R} = 0$

$\therefore$ $\nabla\phi$ is perpendicular to every $\delta\vec{R}$ lying on the surface.

$\nabla\phi$ is normal to the surface at P.

Let $\hat{n}$ be the unit normal at P to the surface, $\Rightarrow$ $\nabla\phi = |\nabla\phi|\hat{n}$

Hence, $\nabla\phi$ is a vector normal to the surface at P and the magnitude is $|\nabla\phi|$

Case II: When P and Q lie on two different level surfaces

then $$\frac{\partial \phi}{\partial n} = \lim_{\delta n \to 0} \frac{\delta \phi}{\delta n} = \lim_{\delta n \to 0} \frac{\nabla \phi \cdot \delta \vec{R}}{\delta n} = |\nabla \phi| \lim_{\delta n \to 0} \frac{|\delta \vec{R}| \cos \theta}{\delta n}$$

Now, $$|\delta \vec{R}| \cos \theta = PQ \cos \theta = \delta n$$

$$\frac{\partial \phi}{\partial n} = |\nabla \phi| \lim_{\delta n \to 0} \frac{|\delta \vec{R}| \cos \theta}{\delta n} = \lim_{\delta n \to 0} |\nabla \phi| \frac{\delta n}{\delta n} = |\nabla \phi|$$

Hence the magnitude of $\nabla \phi$ is the rate of change of ϕ in the direction of the normal to the surface.

4.11.2 Directional Derivative of ϕ in the Direction of $\vec{PQ}$

If $PQ = |\delta \vec{R}|$, δr and $\hat{n}$ is a unit vector normal in the direction of PQ, then the limiting value of $\frac{\delta \phi}{\delta r}$ as $\delta r \to 0$ is known as the directional derivative of ϕ at P along the direction of PQ.

Since $$\delta r = \frac{\delta n}{\cos \theta} = \frac{\delta n}{\hat{n} \cdot \hat{n}'} \text{ and } \delta n = PP'$$

$$\frac{\partial \phi}{\partial r} = \lim_{\delta r \to 0} \frac{\delta \phi}{\delta r} = \lim_{\delta r \to 0} \frac{\nabla \phi \cdot \delta \vec{R}}{\delta r}$$

$$= \lim_{\delta r \to 0} \frac{|\nabla \phi||\delta \vec{R}| \cos \theta}{\delta r} = \lim_{\delta r \to 0} \frac{|\nabla \phi| \delta r \cos \theta}{\delta r}$$

$$= \lim_{\delta r \to 0} |\nabla \phi| \cos \theta \qquad \because \quad \delta r = |\delta \vec{R}|$$

$$= \lim_{\delta r \to 0} |\nabla \phi| \hat{n} \cdot \hat{n}' = \nabla \phi \cdot \hat{n}'$$

which is the directional derivative of ϕ in the direction of $\hat{n}'$. Thus the directional derivative of ϕ in the direction of $\hat{n}'$ is the resolved part of $\nabla \phi$ in the direction of $\hat{n}'$.

Since $$\Delta \phi \cdot \hat{n}' = |\nabla \phi| \cos \theta \le |\nabla \phi|$$

It follows that $|\nabla \phi|$ is maximum in the direction of $\nabla \phi$.

4.11.3 Tangent Planes and Normal Lines

i. **To find the equation of tangent plane:** Let the point of contact of the tangent plane with the given surface be P. Let Q be any point on the tangent plane.

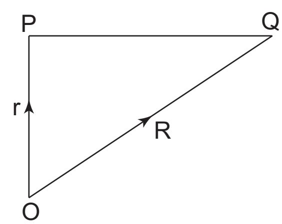


Fig. 4.5

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of P and $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of Q

Then $\vec{PQ} = \vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$ lies on the tangent plane.

Now, as we know that, grad f lies in a direction normal to the tangent plane.

Hence $\vec{R}-\vec{r}$ will be perpendicular to grad f.

($\because$ if a line is perpendicular to a plane then it is perpendicular to every line lying on the plane)

$$\therefore \quad \vec{PQ} \cdot \nabla f = 0$$

$$\therefore \quad \{(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}\} \left\{\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right\} = 0$$

$$\therefore \quad (X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}+(Z-z)\frac{\partial f}{\partial z} = 0$$

which is the equation of the tangent plane at P

ii. To find the equation of normal line: Let P be any point on the normal line of the surface whose position vector is $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Let $\vec{R}$ be the position vector of another point Q on the normal of surface.

Then, $\vec{PQ} = \vec{R}-\vec{r} = (X-x)\,\hat{i} + (Y-y)\,\hat{j} + (Z-z)\,\hat{k}$ also lies on the normal to the surface, as grad f is normal to the surface.

$$\therefore \quad \vec{PQ} \parallel \vec{\nabla} f$$

$$\therefore \quad \vec{PQ}\times\vec{\nabla} f = \vec{0}$$

$$\therefore \quad (\vec{R}-\vec{r})\times\vec{\nabla} f = \vec{0}$$

$$\therefore \quad ((X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}) \times \left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial x}\hat{k}\right) = \vec{0}$$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X-x & Y-y & Z-z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{0}$$

$$\Rightarrow \quad \hat{i}\left((Y-y)\frac{\partial f}{\partial z}-(Z-z)\frac{\partial f}{\partial y}\right)-\hat{j}\left((X-x)\frac{\partial f}{\partial z}-(Z-z)\frac{\partial f}{\partial x}\right)+\hat{k}\left((X-x)\frac{\partial f}{\partial y}-(Y-y)\frac{\partial f}{\partial x}\right) = \vec{0}$$

Comparing components of L.H.S. and R.H.S., we will get

$$\frac{(Y-y)}{\frac{\partial f}{\partial y}} = \frac{(Z-z)}{\frac{\partial f}{\partial z}}, \quad \frac{Z-z}{\frac{\partial f}{\partial z}} = \frac{X-x}{\frac{\partial f}{\partial x}}, \quad \text{and} \quad \frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}}$$

$$\therefore \quad \frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}} = \frac{Z-z}{\frac{\partial f}{\partial z}} \text{ is the equation of normal at } P$$

SOME SOLVED EXAMPLES

Example 4.99. *For $\phi(x, y, z) = x^2y + y^2x + z^2$, find $\nabla\phi$ at the point (1, 1, 1).*

Solution. Here $\qquad \phi = x^2y + y^2x + z^2$

$$\therefore \qquad \frac{\partial\phi}{\partial x} = 2xy + y^2,\ \frac{\partial\phi}{\partial y} = x^2 + 2xy,\ \frac{\partial\phi}{\partial z} = 2z$$

Now
$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \qquad \nabla\phi = (2xy + y^2)\,\hat{i} + (x^2 + 2xy)\,\hat{j} + (2z)\,\hat{k}$$

At (1, 1, 1),
$$\nabla\phi = (2 + 1)\,\hat{i} + (1 + 2)\,\hat{j} + 2\,\hat{k} = 3\hat{i} + 3\hat{j} + 2\hat{k}$$

Example 4.100. If $\phi = \log|\mathbf{r}|$, *then show that grad* $\phi = \dfrac{\vec{r}}{r^2}$.

Solution. Let $|\mathbf{r}| = r$, then $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow \qquad 2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,
$$\frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now
$$\text{grad}\{\log|r|\} = \text{grad}(\log r)$$

$$= \hat{i}\frac{\partial}{\partial x}(\log r) + \hat{j}\frac{\partial}{\partial y}(\log r) + \hat{k}\frac{\partial}{\partial z}(\log r)$$

$$= \hat{i}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(\frac{1}{r}\frac{x}{r}\right) + \hat{j}\left(\frac{1}{r}\frac{y}{r}\right) + \hat{k}\left(\frac{1}{r}\frac{z}{r}\right)$$

$$= \frac{(\hat{i}x + \hat{j}y + \hat{k}z)}{r^2} = \frac{\vec{r}}{r^2}$$

Example 4.101. *Prove that* $\nabla r^n = nr^{n/2}\,\vec{r}$, *when* $r^2 = x^2 + y^2 + z^2$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution. Let $\qquad r^n = (x^2 + y^2 + z^2)^{n/2}$

then,
$$\nabla r^n = \Sigma\,\hat{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{n/2}$$

$$= \Sigma\,\hat{i}\,\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} \times 2x$$

$$= \Sigma n(x^2 + y^2 + z^2)^{\frac{n-2}{2}}\,ix$$

$$= \Sigma nr^{n-2}\,\hat{i}x$$

$$= nr^{n-2}\,\Sigma\hat{i}x = nr^{n-2}\,\vec{r}.$$

Example 4.102. *Find the directional derivative of $f = x^2yz + 4xz^2$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ at the point (1, –2, –1).*

Solution. Unit vector $\hat{n} = \dfrac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \dfrac{[2\hat{i} - \hat{j} - 2\hat{k}]}{3}$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= \hat{i}\,(2xyz + 4z^2) + \hat{j}\,(x^2z) + \hat{k}\,(x^2y + 8xz)$$

The directional derivative of '*f*' in the direction $2\hat{i} - \hat{j} - 2\hat{k}$ at (1, –2, –1) is $\nabla f \,.\, \hat{n}$.

$$= \frac{1}{3}\,[2(2xyz) + 8z^2 + (-x^2z) - 2x^2y - 16xz]$$

$$= \frac{1}{3}\,[4xyz + 8z^2 - x^2z - 2x^2y - 16xz)$$

$$= \frac{1}{3}\,[8 + 8 + 1 + 4 + 16] = \frac{1}{3}\,[37] = \frac{37}{3}.$$

Example 4.103. *The temperature at any point in space is given by T = xy + yz + zx. Determine the directional derivative of T in the direction of the vector $3\hat{i} - 4\hat{k}$ at the point (1, 1, 1).*

Solution. Given, $T = xy + yz + zx$

Thus, $$\nabla T = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)T$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy + yz + zx)$$

$$= \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(x+y)$$

$$\nabla T \text{ at } (1, 1, 1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

Directional derivative of T at (1, 1, 1) in the direction $3\hat{i} - 4\hat{k}$

$$= (2\hat{i} + 2\hat{j} + 2\hat{k})\,.\,\frac{3\hat{i} - 4\hat{k}}{\sqrt{9+16}} = \frac{1}{5}(6-8) = \frac{-2}{5}$$

Example 4.104. *Find the directional derivative of $f(x, y, z) = x^2y^2z^2$ at the point (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = 2 \sin t + 1$ and $z = t - \cos t$ at $t = 0$.*

Solution. Given, $f(x, y, z) = x^2y^2z^2$

$\therefore$ $$\nabla f = 2xy^2z^2\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\hat{k}$$

At $t = 0$, $x = 1$, $y = 1$, $z = 0 - 1 = -1$, *i.e.*, at (1, 1, –1)

Vector normal to the surface $f(x, y, z) = x^2y^2 z^2$ at (1, 1, –1) is $2\hat{i} + 2\hat{j} - 2\hat{k}$

Now $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = e^t\hat{i} + (2\sin t + 1)\hat{j} + (t - \cos t)\hat{k}$$

$$\text{Tangent} = \frac{d\vec{r}}{dt} = e^t\hat{i} + 2\cos t\,\hat{j} + (1+\sin t)\,\hat{k}$$

Tangent at (1, 1, –1) *i.e.*, at $t = 0$ is $\hat{i} + 2\hat{j} + \hat{k}$

Directional derivative in the direction of the tangent is

$$= (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

$$= \frac{2+4-2}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

Example 4.105. *Find the constants 'a' and 'b' so that the surface $ax^2 - byz = (a + 2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).*

Solution. Let $\phi_1 = ax^2 - byz - (a + 2)x = 0$

and $\phi_2 = 4x^2y + z^3 - 4 = 0$

Let n_1, n_2 be the normal vectors to the surface $\phi_1 = 0$, $\phi_2 = 0$

$$\vec{n_1} = \nabla\phi_1 = \hat{i}\,[2ax - (a+2)] + \hat{j}\,[-bz] + \hat{k}\,[-by]$$

and
$$\vec{n_2} = \nabla\phi_2 = \hat{i}\,[8xy] + \hat{j}\,[4x^2] + \hat{k}\,[3z^2]$$

Since the given surfaces are orthogonal at (1, –1, 2) so the angle between the normals $\vec{n_1}$ and $\vec{n_2}$ is 90°.

Therefore
$$\vec{n_1} \cdot \vec{n_2} = 0$$

$$\Rightarrow [(a-2)\hat{i} - 2b\hat{j} + b\hat{k}] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$-8\,(a - 2) - 2b \times 4 + 12b = 0$$

$$-8a + 16 + 4b = 0$$

$$-2a + b + 4 = 0$$

$$2a - b = 4 \qquad \text{...(1)}$$

Since the point (1, –1, 2) lies on the surfaces $ax^2 - byz = (a + 2)x$, we get

$$a + 2b = a + 2 \quad \Rightarrow \quad b = 1$$

Putting $b = 1$ in (1), we get

$$2a - 1 = 4 \quad \Rightarrow \quad 2a = 5 \quad \Rightarrow \quad a = 5/2$$

Hence $a = 2.5$, $b = 1$

Example 4.106. *What is the greatest rate of increase of $u = x^2 + yz^2$ at the point (1, –1, 3).*

Solution. Given, $u = x^2 + yz^2$

then,
$$\nabla u = \hat{i}\frac{\partial}{\partial x}(x^2 + yz^2) + \hat{j}\frac{\partial}{\partial y}(x^2 + yz^2) + \hat{k}\frac{\partial}{\partial z}(x^2 + yz^2)$$

$$= 2x\hat{i} + z^2\hat{j} + 2yz\hat{k}$$

$$= 2\hat{i} + 9\hat{j} - 6\hat{k} \text{ at } (1, -1, 3)$$

Hence the greatest rate of increase of u at $(1, -1, 3) = |\nabla u|$

$$= |2i + 9j - 6k| = \sqrt{4 + 81 + 36} = 11$$

Example 4.107. *Find the equations of the tangent plane and normal to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.*

Solution. The equation of the surface is $f(x, y, z) = 2xz^2 - 3xy - 4x - 7$

then,
$$\nabla f = (2z^2 - 3y - 4)\,\hat{i} - 3x\hat{j} + 4xz\hat{k}$$
$$= 7\hat{i} - 3\hat{j} + 8\hat{k}, \text{ at the point } (1, -1, 2)$$

$\therefore$ $7\hat{i} - 3\hat{j} + 8\hat{k}$ is a vector along the normal to the surface at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$ is $\vec{r} = \hat{i} - \hat{j} + 2\hat{k}$

If $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, then the vector $\vec{R} - \vec{r}$ is perpendicular to the vector ∇f.

$\therefore$ The equation of the tangent plane is $(\vec{R} - \vec{r}) \cdot \nabla f = 0$

i.e., $\{(X\hat{i} + Y\hat{j} + Z\hat{k}) - (\hat{i} - \hat{j} + 2\hat{k})\}\,(7\hat{i} - 3\hat{j} + 8\hat{k}) = 0$

i.e., $\{(X - 1)\hat{i} + (Y + 1)\hat{j} + (Z - 2)\hat{k}\}\,(7\hat{i} - 3\hat{j} + 8\hat{k}) = 0$

i.e., $7(X - 1) - 3(Y + 1) + 8\,(Z - 2) = 0.$

The equations of the normal to the surface at the point $(1, -1, 2)$ are

$$\frac{X - 1}{\frac{\partial f}{\partial x}} = \frac{Y + 1}{\frac{\partial f}{\partial y}} = \frac{Z - 2}{\frac{\partial f}{\partial z}}, \quad i.e., \quad \frac{X - 1}{7} = \frac{Y + 1}{-3} = \frac{Z - 2}{8}.$$

Example 4.108. *Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 25$ at the point $(4, 0, 3)$.*

Solution. Let
$$f = x^2 + y^2 + z^2 - 25 = 0$$

Then
$$\text{grad}\, f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$
$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$
$$= 8\hat{i} + 0\hat{j} + 6\hat{k}, \text{ at the point } (4, 0, 3)$$

Also
$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{r} = 4\hat{i} + 0\hat{j} + 3\hat{k}$$

$\therefore$
$$\vec{R} - \vec{r} = (x - 4)\hat{i} + y\hat{j} + (z - 3)\hat{k}$$

Equation of tangent plane is $(\vec{R} - \vec{r}) \cdot \text{grad}\, f = 0$

$\Rightarrow$ $[(x - 4)\hat{i} + y\hat{j} + (z - 3)\hat{k}] . [8\hat{i} + 6\hat{k}] = 0$

$\Rightarrow$ $8(x - 4) + 6(z - 3) = 0 \quad \Rightarrow \quad 4x + 3z = 25$

The equation of normal line is $(\vec{R} - \vec{r}) \times \text{grad}\, f = 0$

$$\Rightarrow \quad [(x - 4)i + yj + (z - 3)k] \times [8i + 6k] = 0 \quad \Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x - 4 & y & z - 3 \\ 8 & 0 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \qquad 3y\hat{i} + [4(z-3) - 3(x-4)]\hat{j} + (-4y)\hat{k} = 0$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides, we get

$$3y = 0,\ 4(z-3) - 3(x-4) = 0,\ -4y = 0$$

$$\Rightarrow \qquad y = 0,\ \frac{x-4}{4} = \frac{z-3}{3}$$

Required equation of normal is $\dfrac{x-4}{4} = \dfrac{y}{0} = \dfrac{z-3}{3}$

Tangent Planes and Normal Lines

INTERESTING FACT

- The relation between tangent and normal is that they are negative inverse of each other.

REAL LIFE EXAMPLES

- Remember the days of your childhood. You are on a merry-go-round and you are experiencing a force towards outwards. That direction on every single point that pushes you outwards is calculated through this concept.
- You are having a stone attached to a string and you are rotating it in the sky in a circular motion. Consider the string breaks in between, the stone is pushed outwards, and its direction is along the tangent of that circle.
- If we are travelling in a car around a corner and we drive over something slippery on the road (like oil, ice, water or loose gravel) and our car starts to skid, it will continue in a direction tangent to the curve, which is an application of Tangents.
- When you are going fast around a circular track in a car, the force that you feel pushing you outwards is normal to the curve of the road. Interestingly, the force that is making you go around that corner is actually directed towards the centre of the circle, normal to the circle, which is an application of Normal.

4.11.4 Divergence

Let $\vec{F} = F_1(x, y, z)\ \hat{i} + F_2(x, y, z)\ \hat{j} + F_3(x, y, z)\ \hat{k}$ define a vector field, then divergence of a vector field denoted by div $\vec{F}$ and is defined as

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \hat{i} \cdot \frac{\partial \vec{F}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{F}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

If $\qquad \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

then $\qquad \text{div } \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

4.11.5 Curl

The curl of a vector point function $\vec{F}$ is defined as,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}), \text{ then}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Note: curl $\vec{F}$ can also be written as

$$\text{curl } \vec{F} = \sum\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i}$$

Physical Interpretation of Divergence

Consider the motion of the fluid having velocity $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edge $\delta x, \delta y, \delta z$ parallel to the axes in the mass of the fluid with one of its corner at P.

Now the amount of fluid entering the face PB' in the unit time $= V_y\, \delta z\, \delta x$ and the amount of fluid leaving the face $P'B$ in the unit time $= V_{y+\delta y}\, \delta z\, \delta x$.

But $$V_{y+\delta y} = \left[V_y + \frac{\partial V_y}{\partial y}\delta y\right] \text{ nearly}$$

Amount of the fluid leaving the face

$$P'B \text{ in unit time} = \left[V_y + \frac{\partial V_y}{\partial y}\delta y\right]\delta z\, \delta x$$

The net decrease of the amount of fluid due to flow across these two faces $= \dfrac{\partial V_y}{\partial y}\delta x\, \delta y\, \delta z$

Hence the total decrease of fluid inside the parallelopiped in unit time

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x\, \delta y\, \delta z$$

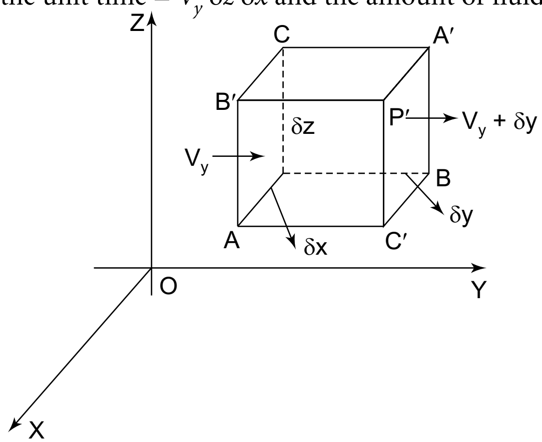


Fig. 4.6

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \text{div } \vec{V}$$

Therefore div $\vec{V}$ gives the rate of such fluid is originating at a point per unit volume.

Similarly if $\vec{V}$ represents an electric flux, div $\vec{V}$ is the amount of flux which diverges per unit volume in unit time. If $\vec{V}$ represents heat flux, div $\vec{V}$ is the rate at which heat is issuing from a point per unit volume. In general, if $\vec{V}$ represents any physical quantity, then div $\vec{V}$ given at each point the rate per unit volume at which the physical quantity is issuing at that point. This explain the justification for the name divergence of a vector point function.

Remark: If div $\vec{V}$ = 0, every where, then such a point function $\vec{V}$ is called a solenoidal vector function.

Physical Interpretation of Curl

Consider the rotation of a rigid body about a fixed axis through O. If ω be its angular velocity, than the velocity $\vec{V}$ of any particle $P(\vec{R})$ of the body is given by,

$$\vec{V} = \vec{\omega} \times \vec{R}$$

if $\quad \vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$

and $\quad \vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

then $\quad \vec{V} = \omega \times \vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$

$$= \hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

$$\text{curl } \vec{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\,(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$$

$$= 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\omega$$

$$\vec{\omega} = \frac{1}{2} \text{curl } \vec{V}$$

Then the angular velocity of rotation at any point is equal to half the curl of its velocity vector point function.

In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Remark: If in a motion, the curl of the velocity vector point function $\vec{V}$ is zero i.e., curl $\vec{V} = 0$, then $\vec{V}$ is said to be irrotational other wise rotational.

4.12 REPEATED OPERATIONS BY ∇

Since grad f and curl $\vec{F}$ are vector point function, we can find their divergence and curl, whereas div $\vec{F}$ being a scalar point function, we can find its gradient only.

Thus we have the following identities.

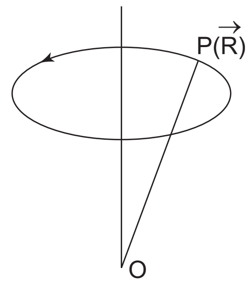


Fig. 4.7

i. div grad $f = \nabla . \nabla f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

ii. curl grad $f = \nabla \times \nabla f = 0$

iii. div curl $\vec{F} = \nabla . \nabla \times \vec{F} = 0$

iv. curl curl $\vec{F}$ = grad div $\vec{F} - \nabla^2 \vec{F}$ *i.e.*, $\nabla \times \nabla \times \vec{F} = \nabla(\nabla . \vec{F}) - \nabla^2 \vec{F}$

v. grad div $\vec{F}$ = curl curl $\vec{F} + \nabla^2 \vec{F}$

i.e., $\nabla(\nabla . \vec{F}) = \nabla \times (\nabla \times \vec{F}) + \nabla^2 \vec{F}$

Proof: i. div grad $f = \nabla . (\nabla f) = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

Now
$$\nabla . \nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$
$$= \nabla^2 f$$

ii.
$$\text{curl grad } f = \nabla \times (\nabla f) = \nabla \times \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$
$$= \Sigma \hat{i} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = 0$$

iii. div curl $\vec{F} = \nabla \cdot \nabla \times \vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}\right)$$

$$= \sum \hat{i} \cdot \left(\hat{i} \times \frac{\partial^2 \vec{F}}{\partial x^2}\right) + \hat{i} \cdot \left(\hat{j} \times \frac{\partial^2 \vec{F}}{\partial x \partial y}\right) + \hat{i} \cdot \left(\hat{k} \times \frac{\partial^2 \vec{F}}{\partial x \partial z}\right)$$

$$= \sum \left[\hat{i} \times \hat{i} \frac{\partial^2 \vec{F}}{\partial x^2} + \hat{i} \times \hat{j} \cdot \frac{\partial^2 \vec{F}}{\partial x \partial y} + \hat{i} \times \hat{k} \frac{\partial^2 F}{\partial x \partial z}\right]$$

$$= \sum \left(\hat{k} \frac{\partial^2 \vec{F}}{\partial x \partial y} - \hat{j} \frac{\partial^2 \vec{F}}{\partial x \partial z}\right) = 0$$

iv. curl curl $\vec{F} = \nabla \times (\nabla \times \vec{F})$

$$= \sum \hat{i} \frac{\partial}{\partial x} \times \left[\hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}\right]$$

$$= \sum \left[\hat{i} \times \hat{i} \times \frac{\partial^2 \vec{F}}{\partial x^2} + \hat{i} \times \hat{j} \times \frac{\partial^2 F}{\partial x \partial y} + \hat{i} \times \hat{k} \times \frac{\partial^2 F}{\partial x \partial z}\right]$$

$$= \sum \left[\left(\hat{i} \frac{\partial^2 F}{\partial x^2}\right)\hat{i} - (\hat{i} \cdot \hat{i}) \frac{\partial^2 F}{\partial x^2} + \left(\hat{i} \cdot \frac{\partial^2 F}{\partial x \partial y}\right)\hat{j} - (\hat{i} \cdot \hat{j}) \frac{\partial^2 F}{\partial x \partial y} + \left(\hat{i} \cdot \frac{\partial^2 F}{\partial x \partial z}\right)\hat{k} - (\hat{i} \cdot \hat{k}) \frac{\partial^2 F}{\partial x \partial z}\right]$$

$$= \sum \hat{i} \frac{\partial}{\partial x}\left[\hat{i} \frac{\partial \vec{F}}{\partial x} + \hat{j} \frac{\partial \vec{F}}{\partial y} + \hat{k} \frac{\partial \vec{F}}{\partial z}\right] - \sum \frac{\partial^2 F}{\partial x^2}$$

$$= \sum \hat{i} \frac{\partial}{\partial x}\left[\hat{i} \frac{\partial \vec{F}}{\partial x} + \hat{j} \frac{\partial \vec{F}}{\partial y} + \hat{k} \frac{\partial \vec{F}}{\partial z}\right] - \nabla^2 F$$

$$= \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F} = \text{grad}(\text{div } \vec{F}) - \nabla^2 \vec{F}$$

v. In the previous problem, we have proved that,

$$\text{curl curl } \vec{F} = \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$\Rightarrow \qquad \nabla (\nabla \cdot \vec{F}) = \text{curl curl } \vec{F} + \nabla^2 \vec{F}$$

$$= \nabla \times (\nabla \times \vec{F}) + \nabla^2 \vec{F}$$

4.13 PROPERTIES OF DIVERGENCE AND CURL

i. $\nabla(fg) = f\nabla g + g\nabla f$

ii. $\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f\nabla \cdot \vec{G}$

iii. $\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f\nabla \times \vec{G}$

iv. $\nabla\ (\vec{F}.\vec{G}) = (\vec{F}.\nabla)\vec{G} + (\vec{G}.\nabla)\vec{F}\ +\ \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F})$

v. $\nabla\ .\ (\vec{F} \times \vec{G}) = \vec{G}.(\nabla \times \vec{F}) - \vec{F}.(\nabla \times \vec{G})$

vi. $\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla.\vec{G}) - \vec{G}(\nabla.\vec{F}) + \vec{G}.\nabla\vec{F} - \vec{F}.\nabla\vec{G}$

Proof: i.

$$\begin{aligned}\nabla(fg) &= \sum \hat{i}\frac{\partial}{\partial x}\ (fg)\\ &= \sum \hat{i}\left\{\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right\}\\ &= \sum \hat{i}\ \frac{\partial f}{\partial x}g\ +\ \sum \hat{i}\ f\ \frac{\partial g}{\partial x}\\ &= g\sum \hat{i}\ \frac{\partial f}{\partial x} = g\,\nabla f + f\,\nabla g\end{aligned}$$

ii.

$$\begin{aligned}\nabla\ .\ (f\vec{G}) &= \sum \hat{i}\frac{\partial}{\partial x}(f\vec{G}) = \sum \hat{i}\left[\frac{\partial f}{\partial x}\vec{G} + \frac{\partial \vec{G}}{\partial x}f\right]\\ &= \sum\ \left(\hat{i}\ \frac{\partial f}{\partial x}\right)\vec{G}\ + \Sigma f\left(\sum \hat{i}\ \frac{\partial \vec{G}}{\partial x}\right)\\ &= \vec{G}.\nabla f + f\nabla.\vec{G}\end{aligned}$$

iii.

$$\begin{aligned}\nabla \times (f\vec{G}) &= \sum \hat{i} \times \frac{\partial}{\partial x}(f\vec{G})\\ &= \sum \hat{i} \times \left(\frac{\partial f}{\partial x}\vec{G} + f\frac{\partial \vec{G}}{\partial x}\right)\\ &= \sum \hat{i}\ \frac{\partial f}{\partial x} \times \vec{G}\ + \Sigma i \times\ \frac{\partial \vec{G}}{\partial x}\ f\\ &= \nabla f \times G + f\nabla \times\ \vec{G}\end{aligned}$$

iv.

$$\nabla\ (\vec{F}.\vec{G}) = \sum \hat{i}\ \frac{\partial}{\partial x}(\vec{F}.\vec{G}) = \sum \hat{i}\ \frac{\partial \vec{F}}{\partial x}.\vec{G} + \sum \hat{i}\ \vec{F}.\frac{\partial \vec{G}}{\partial x}$$

Let us consider

$$\begin{aligned}\vec{F} \times (\nabla \times \vec{G}) &= \vec{F}\ \times\ \sum \hat{i}\ \frac{\partial}{\partial x} \times \vec{G}\\ &= \vec{F} \times \sum \hat{i} \times \frac{\partial \vec{G}}{\partial x}\\ &= \sum \vec{F} \times \hat{i} \times \frac{\partial \vec{G}}{\partial x}\end{aligned}$$

$$= \sum\left[\left(\vec{F}\,.\,\frac{\partial\vec{G}}{\partial x}\right)\hat{i} - (\vec{F}\,.\,\hat{i})\frac{\partial\vec{G}}{\partial x}\right]$$

$$\sum\hat{i}\,\vec{F}\,.\,\frac{\partial\vec{G}}{\partial x} = \vec{F}\times(\nabla\times\vec{G}) + \sum(\vec{F}\,.\,\hat{i})\frac{\partial\vec{G}}{\partial x}$$

Similarly $$\sum\hat{i}\,\frac{\partial\vec{F}}{\partial x}\,.\,\vec{G} = \vec{G}\times(\nabla\times\vec{F}) + \sum(\vec{G}\,.\,\hat{i})\frac{\partial\vec{F}}{\partial x}$$

$\therefore$ $$\nabla(\vec{F}\,.\,\vec{G}) = \vec{F}\times(\vec{\nabla}\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F}) + \sum(\vec{F}\,.\,\hat{i})\frac{\partial\vec{G}}{\partial x} + \sum(\vec{G}\,.\,\hat{i})\frac{\partial\vec{F}}{\partial x}$$

$$= \vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F}) + (\vec{F}\,.\,\nabla)\,\vec{G} + (\vec{G}\,.\,\nabla)\,\vec{F}$$

v. $$\nabla\,.\,(\vec{F}\times\vec{G}) = \vec{G}\,.\,(\nabla\times\vec{F}) + \vec{F}\,(\nabla\times\vec{G})$$

$$\nabla\,.\,(\vec{F}\times\vec{G}) = \sum\hat{i}\,\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum\hat{i}\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum\hat{i}\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$$= \sum\hat{i}\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) - \sum\hat{i}\left(\frac{\partial\vec{G}}{\partial x}\times\vec{F}\right)$$

$$= \sum\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right).\,\vec{G} - \sum\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right).\,\vec{F}$$

$$= \vec{G}\,.\,(\nabla\times\vec{F}) - \vec{F}\,.\,(\nabla\times\vec{G})$$

vi. $$\nabla\times(\vec{F}\times\vec{G}) = \sum\hat{i}\times\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum\hat{i}\times\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \sum\hat{i}\times\vec{F}\times\frac{\partial\vec{G}}{\partial x}$$

$$= \sum(\hat{i}\,.\,\vec{G})\frac{\partial\vec{F}}{\partial x} - \left(\hat{i}\,.\,\frac{\partial\vec{F}}{\partial x}\right)\vec{G} + \left(\hat{i}\,.\,\frac{\partial\vec{G}}{\partial x}\right)\vec{F} - (\hat{i}\,.\,\vec{F})\frac{\partial\vec{G}}{\partial x}$$

$$= \nabla\vec{F}\,.\,\vec{G} - (\nabla\,.\,\vec{F})\,\vec{G} + (\nabla\,.\,\vec{G})\,\vec{F} - \nabla\vec{G}\,.\,\vec{F}$$

$$= \vec{F}(\nabla\,.\,\vec{G}) - \vec{G}\,(\nabla\,.\,\vec{F}) + \vec{G}\,.\,(\nabla\,.\,\vec{F}) - \vec{F}\,.\,(\nabla\,.\,\vec{G})$$

SOME SOLVED EXAMPLES

Example 4.109. *If* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ *and* $r^2 = x^2 + y^2 + z^2$

Prove that i. $\nabla\left(\frac{1}{r^2}\right) = -\frac{2\vec{R}}{r^4}$ ii. $div\ (r^n\vec{R}) = (3+n)\,r^n$

iii. $\nabla\left(\frac{\nabla\cdot\vec{R}}{r}\right) = \frac{-3}{r^3}\vec{R}$

Solution. i.

$$\nabla\left(\frac{1}{r^2}\right) = \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)$$

$$= \sum \hat{i}\frac{-2}{r^3}\left(\frac{\partial r}{\partial x}\right)$$

$$= -2\sum \hat{i}\frac{1}{r^3}\frac{\partial r}{\partial x}$$

$$= -2\sum \hat{i}\frac{1}{r^3}\frac{x}{r} = -2\sum \hat{i}\frac{x}{r^4}$$

$$= \frac{-2}{r^4}\sum (ix) = \frac{-2}{r^4}\vec{R}$$

ii.

$$\text{div}\ (r^n\vec{R}) = r^n\nabla\,.\,\vec{R} + \nabla r^n\,.\,\vec{R}$$

$$= r^n\sum \hat{i}\frac{\partial}{\partial x}(x\hat{i} + y\hat{j} + z\hat{k}) + \nabla r^n\,.\,\vec{R}$$

$$= r^n\,\Sigma\, i\,.\,\hat{i} + \nabla r^n\,\vec{R}$$

$$= r^n\,\Sigma\, 1 + (\nabla r^n)\ \vec{R}$$

$$= 3r^n + (\nabla r^n)\,.\,\vec{R}$$

Now

$$\nabla r^n = \sum \hat{i}\left(\frac{\partial}{\partial x}\right)(r^n)$$

$$= i\, n r^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\, n\, r^{n-1}.\frac{x}{r}$$

$$= \Sigma n r^{n-2}\hat{i}x$$

$$= n r^{n-2}\,\Sigma\,\hat{i}\,x$$

$$= n r^{n-2}\ \vec{R}$$

$$\text{div}\ (r^n\vec{R}) = 3r^n + n r^{n-2}\ \vec{R}\,.\,\vec{R}$$

$$= 3r^n + nr^{n-2} \times r^2$$

$$= (3 + n)r^n$$

iii. $$\nabla\left(\frac{\nabla . R}{r}\right) = \nabla\left[\frac{\sum \hat{i}\frac{\partial}{\partial x}(x\hat{i} + y\hat{j} + z\hat{k})}{r}\right]$$

$$= \nabla\left[\frac{3}{r}\right] = 3\nabla\left[\frac{1}{r}\right] = 3\left[\sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right]$$

$$= 3\,\Sigma\,\hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= \frac{-3}{r^2}\sum\frac{\hat{i}x}{r} = \frac{-3}{r^3}\ \Sigma\hat{i}x$$

$$= \frac{-3}{r^3}\vec{R}$$

Example 4.110. *If* $r^2 = x^2 + y^2 + z^2$ *and* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$, *then show that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Solution. Here, we have

$$\nabla^2 f(r) = \frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2}$$

$$\frac{\partial f(r)}{\partial x} = \frac{\partial}{\partial r}\{f(r)\}\frac{\partial r}{\partial x}$$

$$= f'(r)\,\frac{x}{r} = \frac{f'(r)}{r}\,.x$$

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r).x}{r}\right]$$

$$= \frac{f'(r)}{r} + x.\frac{rf''(r) - f'(r)}{r^2}\frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} + x.\frac{rf''(r) - f'(r)}{r^2}.\frac{x}{r}$$

$$= \frac{f'(r)}{r} + \frac{x^2}{r^3}\ [rf''(r) - f'(r)]$$

Similarly $$\frac{\partial^2 f(r)}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^3}\ [rf''(r) - f'(r)]$$

$$\frac{\partial^2 f(r)}{\partial z^2} = \frac{f'(r)}{r} + \frac{z^2}{r^3}\ [rf''(r) - f'(r)]$$

Thus,

$$\frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2} = \frac{3f'(r)}{r} + \frac{x^2 + y^2 + z^2}{r^3}\ [rf''(r) - f'(r)]$$

$$= \frac{3f'(r)}{r} + \frac{1}{r}\ [rf''(r) - f'(r)]$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= f''(r) + \frac{2f'(r)}{r}$$

Example 4.111. *Prove that div (grad r^n) = $n(n + 1)\ r^{n-2}$.*

Solution. We have, grad $r^n = \nabla r^n = nr^{n-2}\ \vec{R}$

$$\therefore \qquad \text{div (grad } r^n) = \nabla\,.\,(nr^{n-2}\ \vec{R})$$

$$= n[(\nabla r^{n-2})\,.\ \vec{R} + r^{n-2}\,\nabla\,.\ \vec{R}\,]$$

$$= n\ [(n-2)\ (r^{n-4})\ \vec{R}\,.\,\vec{R} + r^{n-2}\,\nabla.\ \vec{R}\,]$$

$$= n\ (n-2)\ (r^{n-4}\,.\,r^2) + nr^{n-2}\,.\,3$$

$$= nr^{n-2}\ (n-2+3) = n\ (n+1)\ r^{n-2}$$

Example 4.112. *Prove that* $\vec{A}\,.\,\nabla\ \vec{B}\,.\,\nabla\,\frac{1}{r} = \frac{3(\vec{A}\,.\,\vec{R})(\vec{B}\,.\,\vec{R})}{r^3} - \frac{\vec{A}\,.\,\vec{B}}{r^3}$,

where $\vec{A}$ *and* $\vec{B}$ *are constant vectors and* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution. We have, $\quad \nabla\frac{1}{r} = -\frac{1}{r^3}\vec{R}$

$$\vec{B}\,.\,\nabla\frac{1}{r} = \frac{-\vec{B}\,.\,\vec{R}}{r^3}$$

$$\nabla\left(\vec{B}\,.\,\nabla\frac{1}{r}\right) = -\nabla\left[\frac{1}{r^3}(\vec{B}\,.\,\vec{R})\right]$$

$$= -\left[\sum \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\,\vec{B}\,.\,\vec{R}\right)\right]$$

$$= -\sum \hat{i}\left[\frac{\partial}{\partial x}\frac{1}{r^3}\,\vec{B}\,.\,\vec{R} + \frac{1}{r^3}\frac{\partial}{\partial x}(\vec{B}\,.\,\vec{R})\right]$$

$$= \sum \hat{i}\left[\left(\frac{-3}{r^4}\cdot\frac{\partial r}{\partial x}\right)(\vec{B}\,.\,\vec{R}) + \frac{1}{r^3}(\vec{B}\,.\,\hat{i})\right]$$

$$\left[\because\ \frac{\partial}{\partial x}(\vec{B}\,.\,\vec{R}) = \frac{\partial\vec{B}}{\partial x}\vec{R} + \vec{B}\,\frac{\partial\vec{R}}{\partial x} = \vec{B}\,\hat{i}\ (\vec{B}\text{ being constant})\right.$$

$$= -\sum \hat{i}\left[\frac{x}{r}\left(\frac{-3}{r^4}\right)(\vec{B}\,.\,\vec{R}) + \frac{1}{r^3}(\vec{B}\,.\,\hat{i})\right]$$

$$= \frac{3}{r^5} \; \Sigma \hat{i} x\,(\vec{B}\,.\,\vec{R}) - \frac{1}{r^3}\,\Sigma(\vec{B}\,.\,\hat{i})\hat{i}$$

$$= \frac{3}{r^5}\,\vec{R}\,(\vec{B}\,.\,\vec{R}) - \frac{1}{r^3}\,\Sigma\hat{i}\,(\vec{B}\,.\,\hat{i})$$

$$= \frac{3}{r^5}\,\vec{R}\,(\vec{B}\,.\,\vec{R}) - \frac{1}{r^3}\,\vec{B}$$

$$= \frac{3}{r^5}\,\vec{R}\,(\vec{B}\,.\,\vec{R}) - \frac{\vec{B}}{r^3}$$

$$\vec{A}\left[\nabla.\left(\vec{B}.\frac{1}{r}\right)\right] = \frac{3}{r^5}(\vec{A}\,.\,\vec{R})(\vec{B}\,.\,\vec{R}) - \frac{\vec{A}\,.\,\vec{B}}{r^3}$$

Example 4.113. *If r is the distance of a point (x, y, z) from the origin. Prove that curl* $\left[k \times grad\,\frac{1}{r}\right]$ + $grad\left[\hat{k}\,.\,grad\,\frac{1}{r}\right] = 0$ *where* $\hat{k}$ *is a unit vector in the direction of z-axis.*

Solution. Since r is the distance of a point (x, y, z) from the origin $r^2 = x^2 + y^2 + z^2$

Now

$$\text{grad}\;\frac{1}{r} = \nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{r}\right)$$

$$= \sum \hat{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)$$

$$= \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{x}{r} = -\sum \hat{i}\,\frac{x}{r^3} = \frac{-\vec{R}}{r^3}$$

$$k \times \text{grad}\;\frac{1}{r} = \frac{-\hat{k}\times\vec{R}}{r^3} = -\frac{\hat{k}\times[x\hat{i} + y\hat{j} + z\hat{k}]}{r^3}$$

$$= -\left[\frac{x\hat{j} - y\hat{i}}{r^3}\right] = \frac{y\hat{i} - x\hat{j}}{r^3}$$

$$\text{curl}\left[k\times\text{grad}\,\frac{1}{r}\right] = \nabla\times\left[\frac{\hat{i}y - \hat{j}x}{r^3}\right]$$

$$= \nabla\times\left[\frac{1}{r^3}(\hat{i}y - \hat{j}x)\right]$$

$$= \frac{1}{r^3}(\nabla\times(y\hat{i} - x\hat{j})] + \nabla\left(\frac{1}{r^3}\right)\times(y\hat{i} - x\hat{j})$$

But

$$\nabla\times(yi - xj) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix}$$

$$= \hat{i}(0+0)+\hat{j}(0-0)+\hat{k}(-1-1)$$

$$= -2\hat{k}$$

$$\text{curl}\left(k\times \text{grad}\,\frac{1}{r}\right) = -\frac{2\hat{k}}{r^3}+\left(\frac{-3}{r^5}\vec{R}\right)\times(y\hat{i}-x\hat{j})$$

$$= -\frac{2\hat{k}}{r^3}-\frac{-3}{r^5}(x\hat{i}+y\hat{j}+z\hat{k})\times(y\hat{i}-x\hat{j})$$

$$= \frac{-2\hat{k}}{r^3}-\frac{3}{r^5}(-x^2\hat{k}-y^2\hat{k}+yz\hat{j}+x\,z\,\hat{i})$$

$$= \left[\frac{-2\hat{k}(x^2+y^2+z^2)+3\hat{k}x^2+3\hat{k}y^2-3yz\hat{j}-3xz\hat{i}}{r^5}\right]$$

$$= \left[\frac{\hat{k}x^2+\hat{k}y^2-2z^2\hat{k}-3yz\hat{j}-3xz\hat{i}}{r^5}\right]$$

or

$$= \left[\frac{x^2\hat{k}+y^2\hat{k}-2z^2\hat{k}-3yz\hat{j}-3xz\hat{i}}{r^5}\right]$$

$$\text{grad}\left(\frac{1}{r}\right) = \left(\frac{-1}{r^3}\vec{R}\right)$$

$$\hat{k}\,.\,\text{grad}\left(\frac{1}{r}\right) = -\frac{\hat{k}\,.\,\vec{R}}{r^3} = \frac{-\hat{k}[x\hat{i}+y\hat{j}+z\hat{k}]}{r^3}$$

$$= \frac{-z}{r^3}$$

$$\text{grad}\left[\hat{k}\,.\,\text{grad}\left(\frac{1}{r}\right)\right] = \nabla\left(\frac{-z}{r^3}\right) = -\nabla\left(\frac{1}{r^3}\right)z = -\frac{1}{r^3}\nabla z$$

$$= \frac{3}{r^5}\vec{R}\,z-\frac{1}{r^3}[\nabla z]$$

$$= \frac{3}{r^5}[x\hat{i}+y\hat{j}+z\hat{k}]z-\frac{1}{r^3}\left[\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right]$$

$$= \frac{3}{r^5}[xz\hat{i}+yz\hat{j}+z^2\hat{k}]-\frac{\hat{k}}{r^3}$$

$$= \frac{3xz\hat{i}}{r^5}+\frac{3yz\hat{j}}{r^5}+\frac{3z^2}{r^5}\hat{k}-\frac{\hat{k}(x^2+y^2+z^2)}{r^5}$$

$$= -\left[\frac{\hat{k}x^2+\hat{k}y^2-2\hat{k}z^2-3xz\hat{i}-3zy\hat{j}}{r^5}\right] \qquad \text{...}(ii)$$

$$\text{grad}\left(\hat{k}\,.\,\text{grad}\,\frac{1}{r}\right)+\text{curl}\left[\hat{k}\times\text{grad}\,\frac{1}{r}\right] = 0 \qquad [\text{from } (i) \text{ and } (ii)]$$

EXERCISE 4.9

1. If $f = 3x^2y - y^3z^2$, find grad f at the point $(1, -2, -1)$.
2. If $\phi = (3r^2 - 4\sqrt{r} + 6r^{-1/3})$, find $\nabla\phi$.
3. If $r = \sqrt{x^2 + y^2 + z^2}$, find grad r.
4. Evaluate grad e^{r^2} where $r^2 = x^2 + y^2 + z^2$.
5. What is the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$?
6. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$.
7. Find the directional derivative of $\phi = 3e^{2x-y+z}$ at the point $A(1, 1, -1)$ in the direction of $\vec{AB}$ where B is the point $(-3, 5, 6)$.
8. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the value of a, b and c.
9. Find the equation of tangent plane and normal to the surface $xyz = 4$ at the point $(1, 2, 2)$.
10. Find the equation of tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at the point $(2, 1, -3)$.
11. Find the divergence and curl of the vectors

    i. $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 + y^2z)\hat{k}$

    ii. $\vec{R} = (x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$
12. Find curl $\vec{F}$ where $\vec{F}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$.
13. Show that each of the vectors

    i. $(x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$

    ii. $3y^4z^2\hat{i} + 4x^2z^2\hat{j} + 3x^2y^2\hat{k}$ are solenoidal
14. If $\vec{A}$ is a constant vector and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

    i. grad $(\vec{A} \cdot \vec{R}) = \vec{A}$

    ii. curl $(\vec{A} \times \vec{R}) = 2\vec{A}$
15. Prove that:

    i. $\nabla \vec{A}^2 = 2(\vec{A} \cdot \vec{V}) = 2\vec{A} \times (\vec{V} \times \vec{A})$ where $\vec{A}$ is a constant vector

    ii. curl $\{(\vec{A} \cdot \vec{R})\vec{R}\} = \vec{A} \times \vec{R}$
16. If 'r' and $\vec{R}$ have their usual meanings and $\vec{A}$ is a constant vector, then prove that

$$\nabla \times \frac{(\vec{A} \times \vec{R})}{r^n} = \frac{2-n}{r^n}\vec{A} + \frac{n(\vec{A} \cdot \vec{R})}{r^{n+2}}\vec{R}$$

17. If V_1 and V_2 be vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2), then prove that,

    i. div $(V_1 \times V_2) = 0$

    ii. grad $(V_1 \cdot V_2) = V_1 + V_2$

    iii. curl $(V_1 \times V_2) = 2(V_1 - V_2)$
18. If $u\vec{F} = \nabla v$, where u, v are scalar fields and $\vec{F}$ is a vector field, then show that $\vec{F}$ curl $\vec{F} = 0$.

Answers

1. $-12\hat{i} - 9\hat{j} - 16\hat{k}$ **2.** $2(3 - r^{-3/2} - r^{-7/3})r$ **3.** $\dfrac{\vec{r}}{r}$ **4.** $2e^{r^2}\vec{r}$

5. $\dfrac{15}{\sqrt{17}}$ **6.** $\dfrac{28}{\sqrt{21}}$ **7.** $\dfrac{-5}{3}$

8. $a = 4k;\ b = -11k,\ c = 10\,k$ where $k = \pm\dfrac{5}{9}$ **9.** $2x + y + z - 6 = 0;\ \dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-2}{1}$

10. $4x + y + z - 6 = 0;\ \dfrac{x-2}{4} = \dfrac{y-1}{1} = \dfrac{z+3}{1}$

11. i. div. $= yz + 3x^2 + 2xz + y^2$ ii. div. $= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

curl $= 2yz\hat{i} + (xy - z^2)\hat{j} + x(6y - z)\hat{k}$ curl $= 0$

12. $\vec{0}$

Review of Vector Calculus

INTERESTING FACT

- Consider a very simple example of a Toilet Pot. The water flowing, which is having a magnitude and direction can teach us all three. The divergence would be in which direction is the water flowing and what is its amount. The gradient would be "at this point of the pot, how fast is the water moving" or "what is the pressure of the water at this point" and the curl would be "How much is the water twisting in this section of the pot".
- Another real case example of gradient is you are moving on a bed. The bed sheet become bumpy at some points. How much it changes is the gradient.

REAL LIFE EXAMPLES

- Einstein law of gravity is a divergence equation.
- They are also fundamental in fluid flow of jet engines and pipelines.

SUBJECTIVE SOLVED QUESTIONS (HOTS)

Example 1. *Divide a number n into three parts x, y, z such that u = ayz + bzx + cxy shall have maximum or minimum and determine what it is.*

Solution. Let n has been divided into three parts x, y, z.

$$x + y + z = n \qquad \text{...(1)}$$

and
$$u = ayz + bzx + cxy \qquad \text{...(2)}$$

For u to be maximum or minimum $du = 0$

$$\Rightarrow \quad (bz + cy)\,dx + (cx + az)\,dy + (ay + bx)\,dz = 0 \qquad \text{...(3)}$$

From (1), on differentiation, $dx + dy + dz = 0$ ...(4)

Multiplying (3) and (4) by '1' and 'λ' respectively, adding and equating to zero the coefficients of dx, dy, dz, we get

$$bz + cy + \lambda = 0 \quad \text{...(5)}$$

$$cx + az + \lambda = 0 \quad \text{...(6)}$$

$$ay + bx + \lambda = 0 \quad \text{...(7)}$$

Multiplying (5) by x, (6) by y (7) by z, and adding, we get

$$2u + n\lambda = 0 \quad \Rightarrow \quad \lambda = -\frac{2u}{n}$$

Now, from (5), (6) and (7), we have

$$0.x + cy + bz - \frac{2u}{n} = 0 \quad \text{...(8)}$$

$$cx + 0.y + az - \frac{2u}{n} = 0 \quad \text{...(9)}$$

and $$bx + ay + 0.z - \frac{2u}{n} = 0 \quad \text{...(10)}$$

Also from (1) $x + y + z - n = 0$ ...(11)

Eliminating x, y, z from (8), (9), (10) and (11), we get

$$\begin{vmatrix} 0 & c & b & \frac{2u}{n} \\ c & 0 & a & \frac{2u}{n} \\ b & a & 0 & \frac{2u}{n} \\ 1 & 1 & 1 & n \end{vmatrix} = 0 \quad \text{...(12)}$$

which gives maximum or minimum values of u. Now to discuss maximum or minimum from (1), let us assume z as a function of x and y, we get

$$\therefore \quad 1 + \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = -1$$

Similarly $$\frac{\partial z}{\partial y} = -1$$

Now $$\frac{\partial u}{\partial x} = bz + cy + (bx + ay)\frac{\partial z}{\partial x}$$

$$= bz + cy - (bx + ay)$$

$$\therefore \quad r = \frac{\partial^2 u}{\partial x^2} = b\frac{\partial z}{\partial x} - b = -2b$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = b\frac{\partial z}{\partial x} + c - a = -b + c - a \text{ or } = c - a - b.$$

and $$t = \frac{\partial^2 u}{\partial y^2} = -2a.$$

we have $$rt - s^2 = (-2b)(-2a) - (c - a - b)^2$$
$$= 2ab + 2bc + 2ac - a^2 - b^2 - c^2$$
$$= a(b + c - a) + b(c + a - b) + c(a + b - c)$$

Let us assume that a, b, c form the three sides of a triangle, then

$b + c - a, c + a - b, a + b - c$ are positive

$\therefore$ $$rt - s^2 = +\text{ve}$$

and $$r = -2b = -\text{ve}.$$

Hence the values of u given by (12) is maximum under the sides of the triangle.

Example 2. *Show that the function $u = \arctan\left(\frac{y}{x}\right)$ satisfies the Laplace equation i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.*

Solution. Here, $$u = \arctan\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

or $$\frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2} \qquad \text{...(3)}$$

Again $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$\Rightarrow$ $$\frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) = -\left[\frac{(x^2 + y^2).0 - y.(2x)}{(x^2 + y^2)^2}\right]$$

$\Rightarrow$ $$\frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2} \qquad \text{...(4)}$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \left[\frac{(x^2 + y^2).0 - x.2y}{(x^2 + y^2)^2}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2} \qquad \text{..(5)}$$

Adding (4) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \textbf{ Proved.}$$

Example 3. *Prove that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2}+\dfrac{\partial^2 u}{\partial \eta^2}$,

where $x = \xi \cos\alpha - \eta \sin\alpha$, $y = \xi \sin a + \eta \cos\alpha$.

Solution. Here

$$x = \xi \cos\alpha - \eta \sin\alpha \qquad \text{...(1)}$$

$$y = \xi \sin\alpha + \eta \cos\alpha \qquad \text{...(2)}$$

From (1) and (2), we have

$$\frac{\partial x}{\partial \xi} = \cos\alpha \text{ and } \frac{\partial x}{\partial \eta} = -\sin\alpha$$

and

$$\frac{\partial y}{\partial \xi} = \sin\alpha \text{ and } \frac{\partial y}{\partial \eta} = \cos\alpha$$

Since u is a function of x and y, where x and y are functions of ξ and η, therefore

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

or

$$\frac{\partial u}{\partial \xi} = \cos\alpha\,.\,\frac{\partial u}{\partial x}+\sin\alpha\,\frac{\partial u}{\partial y} \qquad \text{...(3)}$$

and

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}$$

or

$$\frac{\partial u}{\partial \eta} = -\sin\alpha\,\frac{\partial u}{\partial x}+\cos\alpha\,\frac{\partial u}{\partial y} \qquad \text{...(4)}$$

From (3) and (4), we have operators

$$\frac{\partial}{\partial \xi} = \cos\alpha\,\frac{\partial}{\partial x}+\sin\alpha\,\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = -\sin\alpha\,\frac{\partial}{\partial x}+\cos\alpha\,\frac{\partial}{\partial y}$$

$\therefore$

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)=\left(\cos\alpha\,\frac{\partial}{\partial x}+\sin\alpha\,\frac{\partial}{\partial y}\right).\left(\cos\alpha\,\frac{\partial u}{\partial x}+\sin\alpha\,\frac{\partial u}{\partial y}\right)$$

$$= \cos\alpha\,\frac{\partial}{\partial x}\left(\cos\alpha\,\frac{\partial u}{\partial x}+\sin\alpha\,\frac{\partial u}{\partial y}\right)+\sin\alpha\,\frac{\partial}{\partial y}\left(\cos\alpha\,\frac{\partial u}{\partial x}+\sin\alpha\,\frac{\partial u}{\partial y}\right)$$

i.e.,

$$\frac{\partial^2 u}{\partial \xi^2} = \cos^2\alpha\,\frac{\partial^2 u}{\partial x^2}+2\cos\alpha\sin\alpha\,\frac{\partial^2 u}{\partial x\partial y}+\sin^2\alpha\,\frac{\partial^2 u}{\partial y^2} \qquad \text{...(5)}$$

Similarly

$$\frac{\partial^2 u}{\partial \eta^2} = \sin^2\alpha\,\frac{\partial^2 u}{\partial x^2}-2\sin\alpha\cos\alpha\,\frac{\partial^2 u}{\partial x\partial y}+\cos^2\alpha\,\frac{\partial^2 y}{\partial y^2} \qquad \text{...(6)}$$

Adding (5) and (6), we get

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$ **Proved.**

Note: From the given relations (1) and (2) we can find ξ and η in terms of x and y, then proceeding in the same manner we can show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}.$$

Example 4. *Find the semi vertical angle of the cone of maximum volume and of a given slant height.*

Solution. Let h be the slant height and θ be the semi-vertical angle of cone. Then radius of the base $r = OC = h \sin\theta$ and height of the cone $H = OA = h\cos\theta$

Let V be the volume of the cone, then

$$V = \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi (h\sin\theta)^2 (h\cos\theta)$$

$$= \frac{1}{3}\pi h^3 \sin^2\theta\cos\theta \qquad ...(1)$$

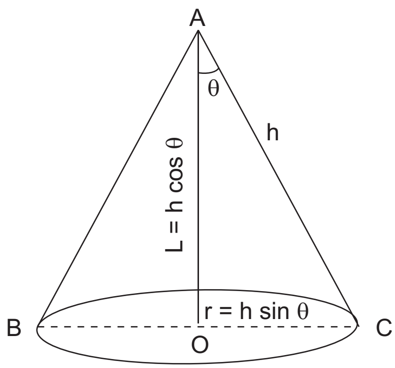


Fig. 4.8

Differentiating (1) w.r.t. θ, we get

$$\frac{dV}{d\theta} = \frac{\pi}{3} h^3[\sin^2\theta(-\sin\theta) + \cos\theta \,.\, 2\sin\theta\cos\theta]$$

$$= \frac{\pi}{3} h^3 \sin\theta\,(2\cos^2\theta - \sin^2\theta)$$

Again, $$\frac{d^2V}{d\theta^2} = \frac{\pi}{3} h^3 [\sin\theta(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta) + (2\cos^2\theta - \sin^2\theta)\,.\,\cos\theta]$$

$$= \frac{\pi}{3} h^3 [-6\sin^2\theta\cos\theta + \cos\theta\,(2\cos^2\theta - \sin^2\theta)]$$

For maximum or minimum, $\frac{dV}{d\theta} = 0$

$\Rightarrow$ $\sin\theta = 0$ or $\tan^2\theta = 2$

$\Rightarrow$ $\theta = 0$ or $\theta = \tan^{-1}\sqrt{2}$

When $\theta = 0$, volume becomes zero and cone becomes a straight line which is not the case.

When $\theta = \tan^{-1}\sqrt{2}$, we have

$$\frac{d^2V}{d\theta^2} = \frac{\pi}{3}h^3\left[-6\,.\,\frac{2}{5}\,.\,\frac{1}{\sqrt{5}}+0\right] = -\text{ve (sign)}$$

Hence the volume of the cone is maximum when $\theta = \tan^{-1}\sqrt{2}$, when $\theta = -\tan^{-1}\sqrt{2}$, volume of cone becomes negative which is meaningless, hence is not the case.

Example 5. *A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring minimum material for its construction.*

Solution. Suppose l be the length 'b' be the breadth and h be the height of the box and S be the surface area and V its volume.

Given that $V = 32$ cc

volume of the box $= l \,.\, b \,.\, h = 32.$

or $$b = \frac{32}{l.h}$$

and $$S = 2(l + b)\, h + lb \quad \text{...(1)}$$

Substituting the value of 'b' in equation (1), we get

$$S = 2\left(l + \frac{32}{lh}\right)h + l\left(\frac{32}{lh}\right)$$

$$\Rightarrow \quad S = 2lh + \frac{64}{l} + \frac{32}{h} \quad \text{...(2)}$$

Now differentiating (2) partially w.r.t. 'l', we get

$$\frac{\partial s}{\partial l} = 2h - \frac{64}{l^2} \quad \text{...(3)}$$

Again differentiating (2) partially w.r.t. 'h', we get

$$\frac{\partial s}{\partial h} = 2l - \frac{32}{h^2} \quad \text{...(4)}$$

For maximum or minimum

$$\frac{\partial s}{\partial l} = 0 \quad \Rightarrow \quad 2h - \frac{64}{l^2} = 0$$

$$\Rightarrow \quad h = \frac{32}{l^2} \quad \text{...(5)}$$

and $$\frac{\partial s}{\partial h} = 0 \quad \Rightarrow \quad 2l - \frac{32}{h^2} = 0 \quad \Rightarrow \quad l = \frac{16}{h^2} \quad \text{...(6)}$$

From equation (5) and (6), $l = 4$, $h = 2$ and $b = 4$

Again $$r = \frac{\partial^2 s}{\partial l^2} = \frac{128}{l^2} = \frac{128}{64} = 2,$$

$$s = \frac{\partial^2 s}{\partial l \partial h} = 2, t = \frac{\partial^2 s}{\partial h^2} = \frac{64}{h^2} = \frac{64}{8} = 8$$

$$\therefore \quad rt - s^2 = \frac{\partial^2 s}{\partial l^2} \cdot \frac{\partial^2 s}{\partial h^2} - \left(\frac{\partial^2 s}{\partial l \partial h}\right)^2$$

$$\therefore \quad 2 \times 8 - (2)^2 = 12$$

So $r = 2$(+ve), so S is minimum for $l = 4$, $b = 4$, $h = 2$.

Example 6. *When travelling x km/h a truck burns diesel oil at the rate of* $\frac{1}{300}\left(\frac{900}{x} + x\right)$ *l/km. If diesel oil costs 40 paise per litre and driver is paid ₹ 1.50 per hour, find the steady speed that will minimize the total cost of trip of 500 km.*

Solution. Velocity of truck $= x$ km/h

Velocity of burning diesel oil $= \frac{1}{300}\left(\frac{900}{x} + x\right)$ l/km

Cost of diesel oil = ₹ 0.40 per litre and
Wages of driver are ₹ 1.50 per hour and
total distance of trip = 500 km

$\therefore$ Total diesel burn in 500 km = $\dfrac{500}{300}\left(\dfrac{900}{x}+x\right)$ litre and its cost = $\dfrac{5}{3}\times\left(\dfrac{900}{x}+x\right)\times 0.40$ rupees.

Time taken in running 500 km = $\dfrac{900}{x}$ hours.

$$\text{Wages of driver} = \frac{500}{x}\times 1.50 = \frac{750}{x} \text{ rupees.}$$

$$\text{Total cost of trip } (c) = \frac{2}{3}\left(\frac{900}{x}+x\right)+\frac{750}{x}$$

or $$c = \frac{1350}{x}+\frac{2}{3}x$$

$\therefore$ $$\frac{dc}{dx} = \frac{-1350}{x^2}+\frac{2}{3} = 0 \text{ (for maximum or minimum)}$$

$\Rightarrow$ $$x^2 = 2025 \quad \Rightarrow \quad x = 45 \text{ km/h}$$

Again $\dfrac{d^2c}{dx^2}$ = +ve. Hence c is minimum when x = 45 km/h.

SUMMARY

1. A function $f(x, y)$ is said to have limit l as $x \to a, y \to b$, if for any arbitrarily chosen positive no. $\varepsilon > 0$, however small, there exists a positive no. $\delta > 0$ such that
$$|f(x, y) - l| < \varepsilon \ \forall\ x, y \text{ for } |x - a| < \delta \text{ and } |y - b| < \delta.$$
2. A function $f(x, y)$ is said to be continuous at\` point (a, b) if $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.
3. If $u = f(x, y)$ is a function of two variables, then its first order partial derivatives w.r.t. x is denoted by $\dfrac{\partial u}{\partial x} = f_x$ and first order partial derivative w.r.t. y is denoted by $\dfrac{\partial u}{\partial y} = f_y$.
4. A function of two variables x, y is said to be a homogeneous function of degree n in x and y if it can be expressible in the form $x^n f(y/x)$.
5. If u is a homogeneous function of degree n in x and y, then by Euler's Theorem,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$$
6. If $u = f(x_1, x_2, ..., x_n)$ and $(x_1, x_2, ..., x_n)$ are functions of 't' only, then $\dfrac{du}{dt}$ is called the total derivative of u w.r.t. t and $\dfrac{du}{dt} = \dfrac{du}{dx_1}\cdot\dfrac{dx_1}{dt}+\dfrac{du}{dx_2}\cdot\dfrac{dx_2}{dt}+\ldots+\dfrac{du}{dx_n}\cdot\dfrac{dx_n}{dt}$.
7. Taylor's Theorem for two variables states that $f(x + h, y + k) = f(x, y) + \left(h\dfrac{\partial}{\partial x}+k\dfrac{\partial}{\partial y}\right) f(x, y) +$
$$\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x, y) + \ldots + \frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y) + \ldots$$

8. If $u_1, u_2, \ldots u_n$ are functions of n independent variables $x_1, x_2, \ldots, x_n$, then the Jacobian of $u_1, u_2, \ldots, u_n$ w.r.t. to $x_1, x_2, \ldots x_n$ is denoted by the symbol $\dfrac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots, x_n)}$ and is defined as

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

9. The term extreme values is used for maximum as well as for minimum values and the points where these values occur are called extreme points.
10. A function $f(x)$ is said to be stationary at $x = a$ if the derivative $f'(x)$ vanishes at $x = a$ *i.e.*, $f'(x) = 0$ at $x = a$
11. If $r > 0$ and $rt - s^2 > 0$, the function $f(x, y)$ has a minimum at given point.
    If $r < 0$ and $rt - s^2 > 0$, the function $f(x, y)$ has a maximum at given point.
    If $rt - s^2 < 0$, the function has neither a maximum nor minimum at given point.
12. For a real valued function $f(x, y, z)$ on IR^3, the gradient $\nabla f(x, y, z)$ is a vector-valued function on IR^3, that is, its value at a point (x, y, z) is the vector

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

13. The divergence of a vector field is the dot product of del operator and the vector field *i.e.*,

$$\text{div } \vec{F} = \nabla . \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

where $\vec{F}(x, y, z) = f_1(x, y, z)\hat{i} + f_2(x, y, z)\hat{j} + f_3(x, y, z)\hat{k}$

14. The curl of a vector field is defined as the cross-product of del operator and vector field *i.e.*,

$$\begin{aligned} \text{curl } \vec{F} &= \nabla \times \vec{F} \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}. \end{aligned}$$

15. Let $F(x, y, z)$ define a surface that is differentiable at a point (x_0, y_0, z_0), then the tangent plane to $F(x, y, z)$ at (x_0, y_0, z_0) is the plane with normal vector $\nabla F(x_0, y_0, z_0)$ that passes through the point (x_0, y_0, z_0). In particular, the equation of tangent plane is

$$\nabla F(x_0, y_0, z_0) . \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

16. Let $F(x, y, z)$ define a surface that is differentiable at a point (x_0, y_0, z_0) then the normal line to $F(x, y, z)$ at (x_0, y_0, z_0) is the line with normal vector $\nabla F(x_0, y_0, z_0)$ that passes through the point (x_0, y_0, z_0). In particular the equation of the normal line is

$$x(t) = x_0 + F_x(x_0, y_0, z_0)t$$
$$y(t) = y_0 + F_y(x_0, y_0, z_0)t$$
$$z(t) = z_0 + F_z(x_0, y_0, z_0)t.$$

OBJECTIVE QUESTIONS

1. Calculate $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^7x^{98} - x^{97}y^8 + x^{105}}{xy^7 + x^8}$

a. does not exist b. 0 c. 1 d. ∞

2. The total derivative is the same as the derivative of the function.

a. True b. False

3. If $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$, then

a. $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exists and non-zero b. $\lim\limits_{(x,\, mx)\to(0,0)} f(x, y)$ exists and zero

c. $\lim\limits_{(x,\, y=x^2)\to(0,0)} f(x, y)$ exists and non-zero d. $\lim\limits_{(x,\, y=2x^2)\to(0,0)} f(x, y)$ exists and non-zero.

4. The existence of first order partial derivatives implies continuity.

a. True b. False

5. If $f(x, y) = \sin(xy) + x^2 \ln(y)$, then find f_{yx} at $(0, \pi/2)$

a. 33 b. 0 c. 3 d. 1

6. Calculate $\dfrac{df}{dt}$ at $t = 1$, when given, $f(x, y) = x^2 + y^3$; $x = t^2 + t^3$; $y = t^3 + t^9$

a. 0 b. 1 c. –1 d. 164

7. Find the maximum directional derivative of the function $f(x, y) = x \ln y + x^2y^2$ at the point (–1, 1).

a. $-2i + j$ b. $15(-2i + j)$ c. 1 d. 5

8. What is the saddle point?

a. Point where function has maximum value.
b. Point where function has minimum value.
c. Point where function has zero value.
d. Point where function neither have maximum value nor minimum value.

9. The minimum value of the given function $xy + a^3\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$ is

a. $3a^2$ b. a^2 c. a d. 1

10. The drawback of Lagrange's Method of Maxima and Minima is?

a. Maxima and Minima is not fixed
b. Nature of stationary points can not be known
c. Accuracy is net good
d. Nature of stationary points is known, but cannot given maxima or minima.

11. In a simple one-constraint Lagrange's multiplier set up, the constraint has to be always one dimension lesser than the objective function

a. True b. False

12. Calculate the curl for the given vector field $\vec{F} = x^3y^2\hat{t} + x^2y^3z^4\hat{j} + x^2z^2\hat{k}$

a. curl $\vec{F} = 0$

b. curl $\vec{F} = (4x^2y^3z^3)\hat{i} + (2xz^2)\hat{j} + (-2xy^3z^4 + 2x^3y)\hat{k}$

c. curl $\vec{F} = (2x^2z^3)\hat{i} - (2yz^2)\hat{j} - (-2x^3y)\hat{k}$

d. curl $\vec{F} = (-4x^2y^3z^3)\hat{i} + (-2xz^2)\hat{j} + (2xy^3z^4 - 2x^3y)\hat{k}$

13. Divergence and curl of a vector field are

a. scalar and scalar b. scalar and vector c. vector and vector d. vector and scalar

14. The curl of a curl of a vector gives a

a. scalar b. vector c. zero value d. cannot say

15. $\nabla \times \nabla \times P$, where P is a vector, is equal to

a. $P \times \nabla \times P - \nabla^2 P$ b. $\nabla^2 P + \nabla(\nabla \times P)$ c. $\nabla^2 P + \nabla \times P$ d. $\nabla(\nabla . P) - \nabla^2 P$

16. For the spherical surface $x^2 + y^2 + z^2 = 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

a. $\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$ b. $\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right)$ c. $\hat{k}$ d. $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

17. The direction of vector A is radially outward from the origin, with $|A| = kr^n$, where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla . A = 0$ is

a. –2 b. 2 c. 1 d. 0

18. Velocity vector of a flow field is given as $\vec{v} = 2xy\,\hat{i} - x^2z\,\hat{j}$. Then curl $\vec{V}$ at (1, 1, 1) is given by

a. $4\hat{i} - \hat{j}$ b. $4\hat{i} - \hat{k}$ c. $\hat{i} - 4\hat{j}$ d. $\hat{i} - 4\hat{k}$

19. Divergence of the three-dimensional radical vector field $\vec{r}$ is given by

a. 3 b. $1/r$ c. $\hat{i} + \hat{j} + \hat{k}$ d. $3(\hat{i} + \hat{j} + \hat{k})$

20. If $\sin^2 y + \cos xy = \pi$, then $\frac{dy}{dx}$ is equal to

a. $\frac{y \sin xy}{\sin 2y - x \sin xy}$ b. $\frac{x \sin xy}{\sin 2y - x \sin xy}$ c. $\frac{y \cos xy}{\cos 2y - x \sin xy}$ d. $\frac{x \cos xy}{\sin 2y + x \cos xy}$

Answers

1. b	**2.** a	**3.** b	**4.** b
5. d	**6.** d	**7.** d	**8.** d
9. a	**10.** b	**11.** b	**12.** d

13. b	**14.** b	**15.** d	**16.** a
17. a	**18.** d	**19.** a	**20.** a

SUBJECTIVE UNSOLVED QUESTIONS (HOTS)

1. Analyse, whether $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2+y^2}$ exists or not, take path along the lines $y = mx$.

2. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

3. Examine the function $\tan^{-1}\left(\dfrac{xy^2}{x+y}\right)$, where it is continuous?

4. If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2\,e^{\theta}\sin\phi$, then show that $\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{\partial^2 u}{\partial\phi^2} = 4xy\,\dfrac{\partial^2 u}{\partial x dy}$.

5. State the continuity assumption for the given $f(x, y)$. Also show that $x^2 f_{11} + xyf_{12} + xyf_{21} + y^2 f_{22} = n(n-1)f$, where $f(x, y)$ is a homogeneous function of degree n.
Here subscripts of 'f' stands for partial derivatives as follows:

$$f_1 = \frac{\partial f}{\partial x}, f_{12} = \frac{\partial^2 f}{\partial x \partial y}, f_{21} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{11} = \frac{\partial^2 f}{\partial x^2}, f_2 = \frac{\partial f}{\partial y}, f_{22} = \frac{\partial^2 f}{\partial y^2}$$

6. The maximum value of $(ax + by + cz)\ e^{-(\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)}$ is $\dfrac{1}{2e}\left[\left(\dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2} + \dfrac{c^2}{\gamma^2}\right)\right]^{1/2}$. Analyze and verify your answer.

7. *ABCD* is a quadrilateral having no re-entrant angle and *P* is a point in its plane. Find the position of *P* for which the sum of the distances from the vertices is minimum.

8. Given $a^x b^y c^z = A$. Find the maximum value of $(x + 1)(y + 1)(z + 1)$. Interpret the result.

9. Show that the vector field $\vec{F} = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$ is irrotational and find its scalar field ϕ so that $\vec{F} = \nabla\phi$.

10. If $\vec{F}$ is a solenoidal vector field, show that curl curl curl curl $\vec{F} = \nabla^4 \vec{F}$

11. Find the value of λ for which the vector $\vec{u} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+\lambda z)\hat{k}$ is a solenoidal vector.

12. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, show that ∇u, ∇v, ∇w are coplanar vectors.

Answers

1. Limit does not exist.

3. The given function continuous on $[(x, y) \in R^2 \,;\, y \neq -x]$

Hint: Take the function as the composition of two functions.

7. P is the point of intersection of the diagonals of the quadrilateral.

8. $\{\log (A\, a\, b\, c)\}^3/(27 \log a \log b \log c)$

11. $\lambda = -2$

12. Hint: Show that $\nabla u \,.\, (\nabla v \times \nabla w) = 0$

PROJECT/ACTIVITIES/PRACTICAL

Model the following problems using MATLAB

1. Draw different surfaces and discuss whether limit exists or not as approaches to the given points. Find the limit, if exists

   i. $f(x) = \dfrac{x-3}{x+5}$ at $x = 5$ ii. $\dfrac{x^2+5}{x+1}$ at $x = 6$.

2. Draw the tangent plane to different surfaces at the given point for $f(x, y) = x^2 + y^2$.
3. Locate the points of relative and absolute extremum for the functions

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

4. Plot the gradient vector fields and level curves for $f(x, y) = xy - \dfrac{x^3}{3}$

ACTIVITY

Identify the points of local maxima, local minima and the points of inflexion of a function of your choice by plotting the graph using graph of paper.

KNOW MORE

1. Calculate $\lim\limits_{(x, y) \to (0,0)} \dfrac{\sin(y)}{x}$

   a. 1 b. 0 c. ∞ d. does not exist

2. If $f(x, y, z, t) = xy + zt + x^2yzt;\ x = k^3;\ y = k^2;\ z = k;\ t = \sqrt{k}$, then, find $\dfrac{df}{dt}$ at $k = 1$

   a. 34 b. 16 c. 32 d. 61

3. Let the temperature at the point (x, y) in a flat plate be given by the function $T(x, y) = 3x^2 + 2xy$. A tub of margarine is placed at $(3, -6)$. Analyse in what direction it should be moved to cool most quickly.

   a. $6i + 6j$ b. $i + j$ c. $-i - j$ d. $6i - 12j$

4. Find the critical points of the function

$$f(x, y) = \frac{\sin^{-1}(y^2).(y^2+3y).(\sin(y^6+7y))}{(y^9+y^{10})} + 10\boldsymbol{x}$$

   a. (0, 0) b. (0, –90) c. (90, 0) d. None exist

5. Find the points on the plane $x + y + z = 9$, which are nearest to origin.

   a. (3, 3, 3) b. (2, 1, 3) c. (2, 2, 2) d. (3, 4, 1)

6. The span of a astroid is increased along both the x and y axes equally. Then the maximum value of $z = x + y$, along the astroid is,

a. increases
b. decreases
c. invariant
d. the scalling of astroid is irrelevent

Hint: Take general form of astroid $x^{2/3} + y^{2/3} = a^{2/3}$, then calculate gradient and equating them with Lagrange's condition.

7. Potential function ϕ is given by $\phi = x^2 - y^2$. What will be the stream function (ψ) with the condition $\psi = 0$ at $x = y = 0$?

a. $2xy$
b. $x^2 + y^2$
c. $x^2 - y^2$
d. $2x^2y^2$

8. The divergence of the vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1, 1, 1) is equal to

a. 7
b. 4
c. 3
d. 0

9. For a scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1, 3) is

a. $\sqrt{\frac{13}{9}}$
b. $\sqrt{\frac{9}{2}}$
c. $\sqrt{5}$
d. $\frac{9}{2}$

Answers

1. d	**2.** c	**3.** a	**4.** d
5. a	**6.** a	**7.** a	**8.** c
9. c			

REFERENCES/SUGGESTED READINGS

1. Bary, N.K. (1964). A Treatise on Trigonometric Series, Pergamon Press.
2. Courant R. (1988). Differential and Integral Calculus, Wiley, New York.
3. Dass, H.K. Advanced Engineering Mathematics, S. Chand Publications.
4. Fleming, W.H. (1965). Functions of Several Variables, Addison Wesley Publishing Company, Reading, MA.
5. Garg, Reena (2019). Engineering Mathematics-I, 2nd Edition, Khanna Book Publishing Co. (P)Ltd.
6. Grewal, B.S. Higher Engineering Mathematics, Khanna Publishers.
7. Jain, R.K.; Iyengar, S.R.K. Advanced Engineering Mathematics, 2nd Edition, Narosa.
8. Knopp, K. (1947) Theory of Functions, Dover, New York.
9. Philip, Franklin (1940). A Treatise on Advanced Calculus, Wiley, Inc. New York.
10. Piskunov, N. (1969). Differential and Integral Calculus, MIR Publishers, Moscow.
11. Prasad, Chandrika (1967). Mathematics for Engineers, Pothishala Private Ltd.
12. Ram, Babu. Engineering Mathematics, Pearson.
13. Thomas, G.B. and Finney, R.L. (1992). Calculus and Analytic Geometry, Addision-Wesley, Reading, MA.
14. Tichmarsh, E.C. (1939). Theory of Functions, Oxford University Press, London.

5 Matrices

UNIT SPECIFICS

This unit incorporates the concepts of inverse and rank of a matrix, rank-nullity theorem, system of linear equations, symmetric, skew-symmetric and orthogonal matrices, determinants, eigen values and eigenvectors, diagonalization of matrices, Cayley-Hamilton Theorem, orthogonal transformation. Interesting facts, video resources, uses of ICT, applications to real life are the wide areas which are included so as to make the topics student friendly and simple.

RATIONALE

Matrices are most widely used to find the solution of system of linear algebraic equations, linear differential equations and non-linear differential equations. Many physical operations such as magnification, rotation and reflection can also be represented mathematically with the help of matrices. These are also used in representing the real world data like traits of people, population, habits etc. Linear systems of equations are used in many areas such as age-problem, speed, time and, many more. The natural frequency of the bridge is the Eigen value of a system having smallest magnitude that models the bridge. The engineers explore this knowledge to ensure the stability of their constructions. Eigen values and eigen vectors allow us to reduce a linear operation in to simple problems. By diagonalization; we can determine the natural frequency of vibrations. Orthogonal transformations are used for computing complex tables into simpler ones.

PRE-REQUISITES

1. Aware from different types of matrices.
2. Basic knowledge of algebra of matrices.
3. Familiar with the concept of adjoint, inverse and determinant with their properties.
4. Know the idea of formation of matrices from the linear system of equations.

UNIT OUTCOMES

After completion of this unit, students will be able to-

U5-01: Recognize consistent and inconsistent systems of linear equations; compute solutions using row echelon form; apply Gauss Elimination, Gauss Jordan methods to find the inverse of a matrix.

U5-02: Solve homogeneous and non-homogeneous system of linear equations; apply rank-nullity theorem in solving complex problems.

U5-03: Compute eigen values and eigen vectors of matrices; learn about Cayley-Hamilton theorem and use it to find the inverse of a square matrix.

U5-04: Familiarize and apply the concept of diagonalisation; find the modal matrix; learn about the conversion of matrices to canonical form from quadratic form.

MAPPING OF UNIT OUTCOMES WITH COURSE OUTCOMES

Unit 5 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (1-Weak Correlation; 2-Medium Correlation; 3-Strong Correlation)				
	CO-1	CO-2	CO-3	CO-4	CO-5
U5-01	–	1	–	1	3
U5-02	2	–	1	–	3
U5-03	2	–	–	1	3
U5-04	–	2	1	1	3

HISTORY

—Augustin Louis Cauchy

Historically, the term matrix was introduced in the 19th-century by English mathematician James Sylvester, but it was his friend Arthur Cayley who developed the algebraic aspect of matrices in two papers in the 1850s. Cayley first applied them to the study of systems of linear equations, which are still very useful. They are also important because, as Cayley recognized, certain sets of matrices form algebraic systems in which many of the ordinary laws of arithmetic (*e.g.*, the associative and distributive laws) are valid but in which other laws (*e.g.*, the commutative law) are not valid. Eigen vectors gradually appeared in 18th century in solving differential equations. It may sound strange but eigen vectors (eigen functions) appeared under various names long before linear algebra, and before the word "vector" came into common use. They played central role in the theory of small oscillations. Cauchy also coined the term racine caractéristique (characteristic root), which is now called eigen value; his term survives in characteristic equation.

The term "eigenvector" comes from German, under the influence of the book of Hilbert-Courant, first edition in 1920s.

INTRODUCTION

The word matrices is plural of the word matrix. Arthur Cayley, first person to introduce the concept of matrices in 1860.

The study of matrices was originated from the idea of various types of linear problems. It has a special relationship with the system of linear equations which occur in many engineering processes. It provides an important list in the study and development of linear algebra. The matrices also occur in presentation of linear system models in applied engineering and control systems. Matrices are widely used in the study of every branch of Mathematics, Science and Engineering.

5.1 DEFINITION

A set of mn numbers (real or complex) arranged in the form of rectangular array having m rows (horizontal lines) and n columns (vertical lines) is called $m \times n$ matrix (read as 'm by n matrix' or 'matrix of order m by n' or 'matrix of type m by n'.

A matrix is generally represented by the symbol $[a_{ij}]$ or (a_{ij}) or $\|a_{ij}\|$.

A matrix is usually denoted by a single capital letter A, B, C etc.

Thus, an $m \times n$ matrix 'A' may be written as

$$A = [a_{ij}]_{m \times n} \quad \text{or} \quad A = [a_{ij}]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{where } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n$$

Each $m \times n$ matrix has $m.n$ number of elements.

Note: Each entry in the matrix is called an element of the matrix.

For example: Let us consider a set of simultaneous system of equations

$$2x + 3y + 3z + 2t = 0$$
$$3x + 2y + 5z + 3t = 0$$
$$4x + 5y + 6z + 7t = 0$$
$$2x + 3y + 4z + 5t = 0$$

Now, we write the coefficients of x, y, z and t of the above equations and enclose them with in the brackets and then, we get

$$A = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 3 & 2 & 5 & 3 \\ 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

The above system of numbers arranged in a rectangular array of rows and columns enclosed by the brackets, is called a matrix.

It has got 4 rows and 4 columns and in all $4 \times 4 = 16$ elements. It is termed as 4×4 matrix.

In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, like a_{ij} lies in i^{th} row and j^{th} column.

5.1.1 Various Types of Matrices

1. **Real Matrix:** A matrix is said to be a real matrix if all its elements are real.

   e.g. $\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 1/2 & 2 & 1 \end{bmatrix}_{2\times 3}$ is a real matrix.

2. **Complex Matrix:** A matrix whose elements may contain complex number is called a complex matrix.

   e.g. $\begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}_{2\times 2}$ is a complex matrix.

3. **Row Matrix:** A matrix which has only one row and any number of columns, is called a row matrix.

   e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times 4}$ is a row matrix.

4. **Column Matrix:** A matrix which has only one column and any number of rows is called a column matrix.

   e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times 1}$ is a column matrix.

5. **Null Matrix or Zero Matrix:** A matrix which has all its elements zero is called a null matrix or zero matrix.

   e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$ are null matrices.

6. **Square Matrix:** A matrix of order $m \times n$ is said to be a square matrix if $m = n$ *i.e.* the number of rows is equal to the number of columns.

   e.g. $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ is a square matrix.

7. **Rectangular Matrix:** A matrix of order $m \times n$ is said to be a rectangular matrix if $m \neq n$ *i.e.* the number of rows is not equal to the number of columns.

   e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}_{2\times 3}$ is a rectangular matrix.

8. **Diagonal Matrix:** A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

   Suppose $A = [a_{ij}]_{n\times n}$ and if $a_{ij} = 0$ for $i \neq j$, then 'A' is diagonal matrix of order $n \times n$.

   Diagonal matrix also written as Diag $\begin{bmatrix} a_{11}, & a_{22}, & \ldots, & a_{nn} \end{bmatrix}$

   e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$ is a diagonal matrix.

9. **Unit Matrix or Identity Matrix:** A square matrix is said to be a unit matrix if all its diagonal elements are unity and non-diagonal elements are zero.

   e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ are unit matrices.

   Unit matrix of order n is also denoted as I_n.

10. **Singular and Non-singular Matrices:** A square matrix 'A' is called a singular matrix if $|A| = 0$ *i.e.* if determinant formed by the elements of 'A' is zero.

e.g. For $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$, we have $|A| = 0$ (singular matrix)

If $|A| \neq 0$, (determinant not equal to zero) then the matrix 'A' is called as non-singular matrix.

e.g. For $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, we have $|A| = -10 \neq 0$ (non-singular matrix)

11. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is called a symmetric matrix if

$$a_{ij} = a_{ji} \ \forall \ i \text{ and } j$$

or $A = A'$ (A' = Transpose of A)

e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & e \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}_{3\times 3}$ are symmetric matrices.

12. **Skew-symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be a skew-symmetric matrix if

a. $a_{ij} = -a_{ji} \quad \forall \ i$ and j or $A = -A'$

b. All diagonal elements are zero

e.g. $\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}_{3\times 3}$ is a skew-symmetric matrix.

13. **Transpose of a Matrix:** Let $A = [a_{ij}]_{m\times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$, then $A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

If $B = [1 \quad 2 \quad 3]$, then $B' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Properties of Transpose of a Matrix: If A' and B' be transposes of A and B respectively then,

a. $(A')' = A$

b. $(A + B)' = A' + B'$

c. $(kA)' = kA'$, k being any number

d. $(AB)' = B'A'$

e. $(ABC)' = C'B'A'$

14. **Orthogonal Matrix:** A square matrix 'A' is called an orthogonal matrix if the product of the matrix 'A' and its transpose A' becomes an identity matrix *i.e.* $A \,.\, A' = I$ where I is an identity matrix.

e.g. For $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, then $A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$, we have

$A \,.\, A' = I$ [Students can verify it at their parts]

then 'A' is an orthogonal matrix.

15. **Triangular Matrix:** A square matrix is said to be a triangular matrix if all the elements lying above or below the leading principal diagonal are zero.

There are two types of triangular matrix.

a. **Upper triangular matrix:** A square matrix all of whose elements below the leading diagonal are zero, is called an upper triangular matrix.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ leading diagonal

is an upper triangular matrix.

b. **Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero, is called a lower triangular matrix.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}$ leading diagonal

is a lower triangular matrix.

16. **Conjugate of a Matrix:**

Let $$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

Conjugate of matrix A is denoted by $\overline{A}$.

$\therefore$ $$\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

Remark: Transpose of the conjugate of a matrix A is denoted by A^{θ}.

Let $$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

$$(\overline{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

or $$A^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

17. **Unitary Matrix:** A square matrix A is said to be unitary if $A^{\theta}A = 1$

e.g.
$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$
$$A^{\theta}A = I.$$

18. **Hermitian Matrix:** A square matrix $A = [a_{ij}]$ is called Hermitian matrix, if every $(ij)^{\text{th}}$ element of 'A' is equal to the conjugate complex $(ji)^{\text{th}}$ element of A.

In other words, $a_{ij} = \bar{a}_{ji}$

e.g.
$$A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$ *i.e.*, conjugate transpose of A.

or $A = (\bar{A})'$.

19. **Skew-Hermitian Matrix:** A square matrix $A = [a_{ij}]$ is called a skew-Hermitian matrix if every $(ij)^{\text{th}}$ element of A is equal to negative conjugate complex of $(ji)^{\text{th}}$ element of A.

In other words, $a_{ij} = -\bar{a}_{ji}$

All the elements in the principal diagonal will be of the form.

$$a_{ii} = -\bar{a}_{ii} \quad \text{or} \quad a_{ii} + \bar{a}_{ii} = 0$$

if $a_{ii} = a + ib$, then $\bar{a}_{ii} = a - ib$

$$(a + ib) + (a - ib) = 0 \quad \text{or} \quad 2a = 0 \quad \text{or} \quad a = 0$$

so a_{ii} is purely imaginary or $a_{ii} = 0$

Hence all the diagonal elements of a skew-Hermitian matrix are either zero or purely imaginary.

e.g.
$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be skew-Hermitian is that,

$$A^{\theta} = -A$$
$$(\bar{A})' = -A$$

20. **Idempotent Matrix:** A matrix, such that $A^2 = A$ is called idempotent matrix.

e.g. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$
$$= A$$

21. **Periodic Matrix:** A matrix A will be called a periodic matrix, if

$$A^{k+1} = A$$

where 'k' is a +ve integer. If k is the least +ve integer, for which $A^{k+1} = A$, then k is said to be period of A. If we choose $k = 1$, we get $A^2 = A$ and we call it to be an idempotent matrix.

22. **Nilpotent Matrix:** A matrix will be called a nilpotent matrix, if $A^k = 0$ (null matrix), where k is a +ve integer; if however k is the least +ve integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

e.g. $$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}, \quad A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

'A' is a nilpotent matrix whose index is 2.

23. **Involutory Matrix:** A matrix 'A' will be called an involutory matrix if $A^2 = I$, unit matrix.

Since $I^2 = I$ (always)

$\therefore$ unit matrix is involutory.

24. **Trace of a Matrix:** Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by Tr (A).

Thus if $A = [a_{ij}]_{n\times n}$, then

$$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$$

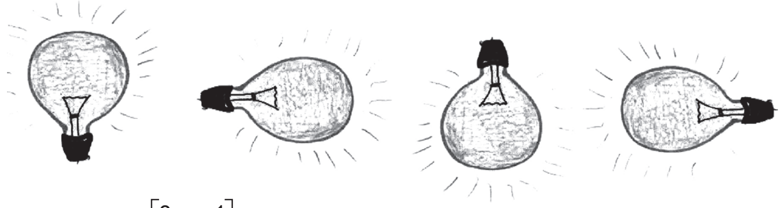
Let $$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, trace $(A) = \text{tr}(A) = 1 + (-3) + 5 = 3$

Properties of Trace of a Matrix: Let A and B be two square matrices of order n and λ be a scalar. Then,

a. $\text{tr}(\lambda A) = \lambda\, \text{tr}\, A$
b. $\text{tr}(A + B) = \text{tr}\, A + \text{tr}\, B$
c. $\text{tr}(AB) = \text{tr}(BA)$

How many matrices does it take to screw in a light bulb?



Just $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, but you might have to apply it repeatedly.

Fig. 5.1

5.1.2 Operation on Matrices

5.1.2.1 Addition of Matrices

The operation 'Addition' on two matrices is performed if they are of same order. Suppose 'A' and 'B' are two matrices of same order, then the sum of these two matrices is obtained by adding the corresponding elements of 'A' and 'B'.

It is denoted as $A + B$

e.g. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3}$, then

$$A + B = \begin{bmatrix} 2+1 & 3+2 & 1+1 \\ 1+1 & 2+1 & 3+1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{2\times3}$$

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$A + B = [a_{ij} + b_{ij}].$$

5.1.2.2 Properties of Matrix Addition

Matrices of the same order only can be added or subtracted.

a. **Commutative law:** Two matrices of same order can be added in any order *i.e.* commutative law holds in matrix addition. If 'A' and 'B' are two matrices, then

$$A + B = B + A \qquad \text{(holds)}$$

(Students can verify it by taking two matrices of same order)

b. **Associative law:** If we have three matrices 'A', 'B' and 'C' of same order, then associativity property holds under addition, *i.e.*,

$$A + (B + C) = (A + B) + C \qquad \text{(students can verify it)}$$

5.1.2.3 Subtraction of Matrices

The operation 'subtraction' on two matrices is performed, if they are of same order.

Suppose 'A' and 'B' are two matrices of same order, then the difference of two matrices is obtained by subtracting each element of the second matrix from the corresponding elements of the first matrix.

It is denoted by $A – B$

e.g. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}_{3\times2}$, then

$$A - B = \begin{bmatrix} 1-1 & 1-1 \\ 1-2 & 2-2 \\ 2-3 & 3-1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 2 \end{bmatrix}_{3\times2}$$

is the difference of two matrices 'A' and 'B'.

5.1.2.4 Scalar Multiplication of a Matrix

If a matrix is multiplied by a scalar quantity k, then each element of it is multiplied by k.

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}_{3\times 3}$, then

$$2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 2 & 4 & 4 \end{bmatrix}_{3\times 3}$$

5.1.2.5 Multiplication of Matrices

Product of two matrices 'A' and 'B' is possible only if the number of columns in 'A' is equal to the number of rows in 'B'.

Let $A = [a_{ij}]_{p\times q}$ and $B = [b_{jk}]_{q\times r}$, then the product AB is defined as

$$C = [c_{ik}]_{p\times r}$$

where $$c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots + a_{in}\, b_{nk}$$

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

and we can write, $$C = AB$$

e.g. a. If we have $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}_{2\times 3}$, then

$$AB = \begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} R_1c_1 & R_1c_2 & R_1c_3 \\ R_2c_1 & R_2c_2 & R_2c_3 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 5 & 3 & 6 \\ 2 & 1 & 2 \end{bmatrix}_{2\times 3}$$

5.1.2.6 Properties of Matrix Multiplication

a. Multiplication of matrices is not commutative *i.e.* If 'A' and 'B' are two matrices, then

$$AB \neq BA \qquad \text{(need not to be equal)}$$

b. Matrix multiplication is associative, if confirmability is assured, *i.e.* for three matrices 'A', 'B' and 'C', we have

$$A(BC) = (AB)C$$

c. Matrix multiplication is distributive with respect to addition. For three matrices A, B and C, we have

$$A(B + C) = AB + AC$$

d. Multiplication of a matrix 'A' by a unit matrix 'I' is a matrix 'A' itself. *i.e.*

$$AI = IA = A$$

e. Multiplicative inverse of a matrix 'A' exists if $| A | \neq 0$ *i.e.*

$$AA^{-1} = A^{-1}A = I$$

For example: If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2}$

then obtain the product AB and explain why BA is not defined?

Solution. Since the number of columns of A (3×3) = number of rows in B (3×2)

therefore the product AB is defined

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times3} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}_{3\times2}$$

Now, for BA, the number of columns of B $(3 \times 2) \neq$ the number of rows of A (3×3) therefore, the product BA is not defined.

> **Note:** 1. If A and B are two matrices of order $m \times n$ and $p \times q$ respectively, then the product AB is possible only when $n = p$ and the order of AB is $m \times q$.
>
> 2. For two matrices 'A' and 'B' if the product AB exist, then the product BA may or may not exist.

Pictorial Representation

1. Step by step visualization of matrix multiplication: http://matrixmultiplication.xyz

EXERCISE 5.1

1. Given $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then find

a. $2A + 3B$ b. $3A - 4B$

2. Two matrices A and B are such that $3A - 2B = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ and $-4A + B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$, then find A and B.

3. If A = diag. [2, 9, 4] and B = diag. [–3, 7, 6], then find

a. $A + B$ b. $A - B$ c. $7A + 2B$ d. $9A - 11B$

4. Given $A = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$, then find the matrix X such that $2A + 3X = 5B$.

5. Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, then find

a. AB b. BA

Also show that $AB \neq BA$.

Answers

1. a. $\begin{bmatrix} 3 & 10 & 3 \\ 8 & 3 & 6 \\ 2 & 2 & 13 \end{bmatrix}$ b. $\begin{bmatrix} -4 & -2 & -4 \\ -5 & -4 & 9 \\ 3 & 3 & -6 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$

3. a. diag. [–1, 16, 10] b. diag. [5, 2, –2]
c. diag. [8, 77, 40] d. diag. [51, 4, –30]

4. $X = \begin{bmatrix} 12 & 4/3 \\ 4 & -14/3 \\ 25/3 & 28/3 \end{bmatrix}$ 5. a. $\begin{bmatrix} 3 & 12 & 11 \\ 4 & 13 & 8 \\ 0 & -1 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 11 & 9 & 13 \\ 3 & 2 & 4 \\ 0 & 5 & 8 \end{bmatrix}$

5.2 ELEMENTARY OPERATIONS (TRANSFORMATION)

Any one of the following operations on a matrix is called an elementary transformation or E-transformation.

1. Interchange of any two rows (or columns). The interchange of i^{th} and j^{th} rows is denoted by R_{ij} or $R_i \leftrightarrow R_j$.
   Similarly, the interchange of i^{th} and j^{th} columns is denoted by C_{ij} or $C_i \leftrightarrow C_J$.
2. Multiplication of the elements of any row (or column) by a non-zero scalar quantity. The multiplication of i^{th} row by k is denoted by kR_i.
   Similarly, the multiplication of i^{th} column by k is denoted by kC_i.
3. Addition of constant multiplication of the elements of any row (or column) to the corresponding elements of any other row (or column).

5.2.1 Elementary Matrix

A matrix obtained from the unit matrix by applying any of the elementary transformation is called an elementary matrix.

e.g. Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_3$, then we have
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{ is an elementary matrix.}$$

5.3 ECHELON FORM OF A MATRIX

5.3.1 Row-Echelon Form of a Matrix

A matrix is said to be in row-echelon form if

a. All zero rows, if any, are at the bottom of the matrix.

b. First non-zero element of every row is on the right hand side of the first non-zero element in the preceeding row.

Note: The first non-zero element of any row is called key-element or pivotal element or pivot of that row.

5.3.2 Row Reduced Echelon Form of a Matrix

A matrix is said to be in row reduced echelon form if

a. It is in row echelon form.
b. Every key element is unity.
c. The elements above the key element in every column are all zero.

5.3.3 Column Echelon Form of a Matrix

A matrix is said to be in column echelon form if

a. All zero columns, if any, are at the extreme right of the matrix.
b. First non-zero element of every column is below the first non-zero element in the preceeding column.

Remark: The first non-zero element of any column is called the key-element or pivotal element or pivot of that column.

5.3.4 Column Reduced Echelon Form of a Matrix

A matrix is said to be in column reduced echelon form if

a. It is in column echelon form.
b. Every key element is unity.
c. The elements to the left of the key element in every row are all zero.

Remark: If A is row echelon form, then its transpose A' is in the column echelon form.

5.4 DETERMINANTS

The theory of determinant was originated from the study of system of linear equations. The determinant is a scalar value which is a function of the entries of a square matrix. The determinant of a square matrix $A = [a_{ij}]$ is denoted by det A or det (A) or $|A|$.

The determinant of a 1×1 (read as one cross one) matrix $A = [a]$ is denoted as $|A| = a$ and is called the determinant of order one.

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is called the determinant of order two.

Similarly, the determinant of a 3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is denoted as $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and is called the determinant of order 3.

5.4.1 Explanation of Determinant of Order Two (or Second Order)

Consider the following system of two linear equations having two unknowns x and y

$$a_1x + b_1y = 0 \qquad ...(1)$$

$$a_2x + b_2y = 0 \qquad ...(2)$$

Equation (1) gives

$$\frac{x}{y} = -\frac{b_1}{a_1} \qquad ...(3)$$

Equation (2) gives $\dfrac{x}{y} = -\dfrac{b_2}{a_2}$ ...(4)

From equations (3) and (4), we can eliminate x and y to get

$$-\frac{b_1}{a_1} = -\frac{b_2}{a_2}$$

$\Rightarrow$ $a_1b_2 - b_1a_2 = 0$

The number $a_1b_2 - b_1a_2$ is called the determinant of order two.

The number $a_1b_2 - a_2b_1$ is also represented more conveniently by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

The numbers a_1, a_2, b_1, b_2 are called the elements of the determinant.

> **Remark:** To expand a determinant of order 2 × 2, we apply the rule of cross-multiplication. The value of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

SOME SOLVED EXAMPLES

Example 5.1. *Evaluate the determinant of* $\begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$.

Solution. Let $A = \begin{bmatrix} 5 & -2 \\ 3 & 7 \end{bmatrix}$

then $|A| = \begin{vmatrix} 5 & -2 \\ 3 & 7 \end{vmatrix}$

$= 35 + 6 = 41$ **Answer**

Example 5.2. *Find the value of x, if* $\begin{vmatrix} x & 3 \\ 6 & 2x \end{vmatrix} = \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix}$.

Solution. Given $\begin{vmatrix} x & 3 \\ 6 & 2x \end{vmatrix} = \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix}$

then $2x^2 - 18 = 18 - 18$

$\Rightarrow$ $2x^2 - 18 = 0$

$$\Rightarrow \qquad 2x^2 = 18$$
$$\Rightarrow \qquad x^2 = 9$$
$$\Rightarrow \qquad x = \pm 3 \quad \textbf{Answer}$$

5.4.2 Expansion of Determinant of Third Order

Consider the following system of three linear equations having three unknowns x, y and z.

$$a_1x + b_1y + c_1z = 0 \qquad \text{...(5)}$$
$$a_2x + b_2y + c_2z = 0 \qquad \text{...(6)}$$
$$a_3x + b_3y + c_3z = 0 \qquad \text{...(7)}$$

To eliminate x, y, z from the above set of three equations, we solve (6) and (7)

$$\frac{x}{b_2c_3 - c_2b_3} = \frac{y}{c_2a_3 - a_2c_3} = \frac{z}{a_2b_3 - b_2a_3} = k \quad \text{(say)}$$

From here, we have $\qquad x = k(b_2c_3 - c_2b_3)$

$$y = k(c_2a_3 - a_2c_3)$$
$$z = k(a_2b_3 - b_2a_3)$$

Substituting these values of x, y, z in equation (5), we get

$$a_1(b_2c_3 - c_2b_3)k + b_1(c_2a_3 - a_2c_3)k + c_1(a_2b_3 - b_2a_3)k = 0$$

or $\qquad a_1(b_2c_3 - c_2b_3) + b_1(c_2a_3 - a_2c_3) + c_1(a_2b_3 - b_2a_3) = 0 \qquad$ ...(8)

(Students think why k cann't be zero)

This expression on the left hand side of (8) is called the determinant of third order. Symbolically, it is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Rightarrow \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - c_2b_3) + b_1(c_2a_3 - a_2c_3) + c_1(a_2b_3 - b_2a_3)$$

$$\text{or} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Remarks: i. The above expansion of determinant is known as expanding in terms of 1st row.

Similarly, the determinant can be expanded along any row or any column and in each case the value of the determinant remains the same.

ii. Sign before each term = $(-1)^{i+j}$, where 'i' and 'j' indicate the row and the column in which the element lie.

This is valid for determinant of any order.

SOME SOLVED EXAMPLES

Example 5.3. *Evaluate the determinant of* $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{bmatrix}$.

Solution. Given $$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$$

Expanding along 1st row, we have

$$|A| = 1\begin{vmatrix} 6 & 10 \\ 11 & 38 \end{vmatrix} - 3\begin{vmatrix} 2 & 10 \\ 31 & 38 \end{vmatrix} + 5\begin{vmatrix} 2 & 6 \\ 31 & 11 \end{vmatrix}$$

$$= 1(228 - 110) - 3(76 - 310) + 5(22 - 186)$$

$$= 1(118) - 3(-234) + 5(-164) = 118 + 702 - 820$$

$$= 0$$ **Answer**

Example 5.4. *If* $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, *then show that* $|3A| = 27|A|$.

Solution. Given $$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

then $$3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

then the determinant of $|3A|$,

$$|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

Expanding along 1st row, we have

$$|3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 6 \\ 0 & 12 \end{vmatrix} + 3\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$$

$$= 3(36 - 0) - 0(0 - 0) + 3(0 - 0)$$
$$= 108 - 0 + 0$$
$$= 108 \quad \text{(L.H.S.)}$$

and $$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

Expanding along 1st row, we have

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$
$$= 1(4) - 0 + 1(0)$$
$$= 4$$

then $$27\,|A| = 27 \times 4 = 108 \quad \text{(R.H.S.)}$$

Thus, we have, $$|3A| = 27\,|A| \quad \textbf{Proved}$$

5.4.3 Properties of Determinant

Property i. The value of the determinant remains unchanged if all it rows are changed into columns and all columns are changed into rows.

Example 5.5. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

Solution. Given, $$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Expanding along second row, we have

$$\Delta = 36 \qquad \text{(Students can find)}$$

Interchanging rows and columns, then we have

$$\text{(say) } \Delta_1 = \begin{vmatrix} 1 & 0 & 7 \\ 2 & 0 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

Expanding along first row, we have

$$\Delta_1 = 36 \qquad \text{(Students can verify it)}$$

Thus, $$\Delta = \Delta_1$$

Property ii. If any two rows or any two columns of a determinant are interchanged, then the sign of the value of the determinant is also changed.

Example 5.6. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 6 \\ 1 & 2 & 3 \end{vmatrix}$.

Solution. Given, $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 6 \\ 1 & 2 & 3 \end{vmatrix}$

After expanding along 1st row, we have

$$\Delta = 2$$

Using Property ii, *i.e.* interchanging second and third rows, we have

$$\text{(say) } \Delta_1 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 6 \end{vmatrix}$$

then expanding through 1st row, we have

$$\Delta_1 = -2$$

Thus, we have, $\Delta = -\Delta_1$

Property iii. If any two rows or two columns of a determinant are identical, then the value of the determinant is always zero.

Example 5.7. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix}$.

Solution. Given $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix}$

Expanding along 1st row, we have

$$\Delta = 0 \qquad \text{(Students can calculate)}$$

Thus, $\Delta = 0$

Property iv. If each element of any row or column of a determinant is multiplied by the same constant, then the value of the determinant is also multiplied by that factor.

Example 5.8. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$.

Solution. Given $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$

Expanding through 1st row, we have

$$\Delta = -3$$

Multiply the 1st row by 5, we get

$$\text{(say) } \Delta_1 = \begin{vmatrix} 5 & 10 & 15 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= -15 = 5(\Delta)$$

$$\Rightarrow \qquad \Delta_1 = 5\Delta$$

Property v. The value of the determinant remains unchanged if the elements of one row (or column) be added to any constant multiple of the corresponding elements of other row (or column) respectively.

Example 5.9. *Evaluate* $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 0 & 4 & 6 \end{vmatrix}$.

Solution. Given $$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$$

On expanding through 1st row, we have

$$\Delta = -2$$

On multiplying the second column by 3 and adding to the first column, we get

$$\text{(say) } \Delta_1 = \begin{vmatrix} 1+6 & 2 & 4 \\ 3+3 & 1 & 5 \\ 0+12 & 4 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 2 & 4 \\ 6 & 1 & 5 \\ 12 & 4 & 6 \end{vmatrix}$$

On expanding along 1st row, we have

$$\Delta_1 = -2$$

Thus $$\Delta = \Delta_1$$

Property vi. If each element of a row (or column) of a determinant is expressed as a sum of two (or more) terms, then the determinant can also be expressed as the sum of two (or more) determinant.

Example 5.10. *Evaluate* $\Delta = \begin{vmatrix} 2+1 & 1 & 0 \\ 3+1 & 0 & 1 \\ 2+2 & 1 & 0 \end{vmatrix}$.

Solution. Given $$\Delta = \begin{vmatrix} 2+1 & 1 & 0 \\ 3+1 & 0 & 1 \\ 2+2 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 1$$ (Students can calculate and verify)

Remark: If A is a square matrix of order n, then $|kA| = k^n |A|$

Example 5.11. *Using Properties of determinant, prove that*

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)\,(b-c)\,(c-a)\,(ab+bc+ca).$$

Solution. Let

$$\Delta = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we have

$$\Delta = \begin{vmatrix} a & a^2 & bc \\ b-a & b^2-a^2 & ca-bc \\ c-a & c^2-a^2 & ab-bc \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a & a^2 & bc \\ b-a & (b-a)(b+a) & c(a-b) \\ c-a & (c-a)(c+a) & b(a-c) \end{vmatrix}$$

Taking out $(b-a)$ and $(c-a)$ common from R_2 and R_3 respectively, we have

$$\Delta = (b-a)\,(c-a)\begin{vmatrix} a & a^2 & bc \\ 1 & a+b & -c \\ 1 & a+c & -b \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$, we have

$$\Delta = (b-a)\,(c-a)\begin{vmatrix} a & a^2 & bc \\ 1 & a+b & -c \\ 0 & c-b & c-b \end{vmatrix}$$

Taking out $(c-b)$ common from R_3, we have

$$= (b-a)\,(c-a)\,(c-b)\begin{vmatrix} a & a^2 & bc \\ 1 & a+b & -c \\ 0 & 1 & 1 \end{vmatrix}$$

Operating through third row, we have

$$= [(b-a)\,(c-a)\,(c-b)]\,(-1)\begin{vmatrix} a & bc-a^2 \\ 1 & -c-a-b \end{vmatrix}$$

$$= [(b-a)\,(c-a)\,(c-b)]\,(-1)\,[-ac-a^2-ab-bc+a^2]$$

$$= [(b-a)\,(c-a)\,(c-b)]\,(-1)\,[(-1)\,(ac+ab+bc)]$$

$$= (a-b)\,(b-c)\,(c-a)\,(ab+bc+ca) \quad \textbf{Proved}$$

Example 5.12. *Show that* $\Delta = \begin{vmatrix} 1 & a & abc \\ 1 & b & bca \\ 1 & c & cab \end{vmatrix} = 0.$

Solution. Let
$$\Delta = \begin{vmatrix} 1 & a & abc \\ 1 & b & bca \\ 1 & c & cab \end{vmatrix}$$
Taking out common *abc* from c_3, we get
$$\Delta = abc \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$
$$= 0 \qquad \text{[As 1st and 3rd column are same] [As per Property (iii)]}$$

Example 5.13. *Using Property of determinant, prove that* $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x+2)\,(x-1)^2.$

Solution. Given
$$\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$
Operating $C_1 \to C_1 + C_2 + C_3$, we get
$$\Delta = \begin{vmatrix} x+2 & 1 & 1 \\ x+2 & x & 1 \\ x+2 & 1 & x \end{vmatrix}$$
Taking out $(x + 2)$ common from C_1, we get
$$= (x+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$
Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have
$$\Delta = (x+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & (x-1) & 0 \\ 0 & 0 & (x-1) \end{vmatrix}$$
Taking out $(x - 1)$ common from R_2 and R_3 respectively, we have
$$\Delta = (x+2)\,(x-1)^2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
Expanding along R_2, we have
$$\Delta = (x+2)\,(x-1)^2 \,.\, (1) \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (x + 2)\,(x - 1)^2 \,.\, (1)$$
$$= (x + 2)\,(x - 1)^2 \quad \textbf{Proved}$$

EXERCISE 5.2

1. Evaluate the following determinant:

a. $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$ b. $\begin{vmatrix} \sqrt{6} & \sqrt{5} \\ \sqrt{20} & \sqrt{24} \end{vmatrix}$

c. $\begin{vmatrix} 210 & 117 & 345 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$ d. $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

2. Evaluate the given determinant $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

a. with the help of second row b. with the help of third column

3. Find the value of the given determinant $\Delta = \begin{vmatrix} 0 & 1 & \sec\theta \\ \tan\theta & -\sec\theta & \tan\theta \\ 1 & 1 & 1 \end{vmatrix}$.

4. Using the Properties of determinant, prove the following:

a. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b - c)\,(c - a)\,(a - b)\,(a + b + c)$ b. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

c. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2$ d. $\begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

e. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(a + b + c)\,(ab + bc + ca - a^2 - b^2 - c^2)$

5. Show that $x = 2$ is a root of the given equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x - 3 \\ -3 & 2x & x + 2 \end{vmatrix} = 0$ and solve it completely.

6. If a, b and c are real and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, then show that either $a + b + c = 0$ or $a = b = c$.

Answers

1. a. $x^3 - x^2 + 2$ b. 2 c. 2691 d. 40

2. a. 23 b. 23 **3.** $\sec\theta(\sec\theta + \tan\theta)$

5.4.4 Applications of Determinants

5.4.4.1 Area of Triangle by using Determinant

The area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\text{Area} = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

or $$\frac{1}{2}\left[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right]$$

This expression in the form of determinant can be written as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{or} \quad \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

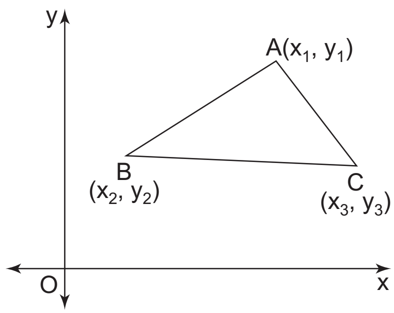


Fig. 5.2

> **Note:** 1. Determinant could be negative but area is always non-negative *i.e.*, ≥ 0.
> 2. If the area of a triangle is given, then use positive as well as negative values for calculations.
> 3. Three points are **collinear** if and only if area of triangle formed by three points is zero.

SOME SOLVED EXAMPLES

Example 5.14. *Find the area of the triangle whose vertices are (2, 7), (1, 1), (10, 8).*

Solution. Area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Vertices of triangle are $(\underset{x_1}{2}, \underset{y_1}{7}), (\underset{x_2}{1}, \underset{y_2}{1}), (\underset{x_3}{10}, \underset{y_3}{8})$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 10 \\ 7 & 1 & 8 \end{vmatrix}$$

Expanding about R_1, $$\Delta = \frac{1}{2}\left[1(8-10) - 1(16-70) + 1(2-7)\right]$$

$$= \frac{1}{2}\left[1(-2) - 1(-54) + 1(-5)\right]$$

$$= \frac{1}{2}[-2+54-5] = \frac{47}{2}$$

Thus, required area of triangle is $\frac{47}{2}$ square unit.

Example 5.15. *Find the area of triangle whose vertices are (–2, –3), (3, 2), (–1, –8).*

Solution. Vertices of triangle are $(\underset{x_1}{\underset{\downarrow}{-2}}, \underset{y_1}{\underset{\downarrow}{-3}}), (\underset{x_2}{\underset{\downarrow}{3}}, \underset{y_2}{\underset{\downarrow}{2}}), (\underset{x_3}{\underset{\downarrow}{-1}}, \underset{y_3}{\underset{\downarrow}{-8}})$

Area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & -1 \\ -3 & 2 & -8 \end{vmatrix}$$

$$= \frac{1}{2}[1(-24+2)-1(16-3)+1(-4+9)] \qquad \text{[Expanding about } R_1\text{]}$$

$$= \frac{1}{2}[-22-13+5] = -\frac{30}{2} = -15$$

Since area of triangle is always non-negative, therefore, the required area of triangle is 15 square units.

Example 5.16. *Find the value of k if area of triangle is 4 square units and vertices are (–2, 0), (0, 4), (0, k).*

Solution. Area of the triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Here, area of triangle = 4 square units

$\therefore$ $\Delta = \pm 4$

[area is non-negative but determinant can be positive and negative]

Putting $x_1 = -2, x_2 = 0, x_3 = 0$ and $y_1 = 0, y_2 = 4, y_3 = k$

$$\pm 4 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 0 & 4 & k \end{vmatrix}$$

$$\pm 4 = \frac{1}{2}[-(-2)(k-4)] \qquad \text{[Expanding about } R_2\text{]}$$

$$\pm 8 = 2(k-4)$$

$$\pm 4 = k-4$$

So, $4 = k - 4$ and $-4 = k - 4$

$$4 + 4 = k \qquad -4 + 4 = k$$
$$8 = k \qquad 0 = k$$
$$\therefore \quad k = 8 \qquad \therefore \quad k = 0$$

Required value of $k = 8, 0$.

Example 5.17. *Show that the points (–2, –1), (7, 8), (–3, –2) are collinear.*

Solution. Three points are collinear if they lie on same line.

$\therefore$ Area of triangle = 0

Area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \qquad ...(1)$$

Points are $(\underset{x_1}{-2}, \underset{y_1}{-1}), (\underset{x_2}{7}, \underset{y_2}{8}), (\underset{x_3}{-3}, \underset{y_3}{-2})$

Putting the values in (1), we get

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & 7 & -3 \\ -1 & 8 & -2 \end{vmatrix}$$

Expanding about R_1, $\Delta = \frac{1}{2}[1(-14+24)-1(4-3)+1(-16+7)]$

$$= \frac{1}{2}[10-1-9] = 0$$

So, $\Delta = 0$

Hence, given points are collinear.

EXERCISE 5.3

1. Find the area of the triangle with vertices at the point given in each of the following:
   a. (1, 0), (6, 0), (4, 3)
   b. (3, 8), (–4, 2), (5, 1)
2. Find the value of k in following if
   i. area of triangle is 4 sq. units and vertices are $(k, 0)$, (4, 0), (0, 2).
   ii. area of triangle is 35 sq. units and vertices are (2, –6), (5, 4), $(k, 4)$.
3. Show that points $A(a, b + c)$, $B(b, c + a)$, $C(c, a + b)$ are collinear.

Answers

1. a. 15/2 b. 61/2
2. i. 0, 8 ii. 12, –2

5.4.5 Minors and co-factors

5.4.5.1 Minors

Let $A = [a_{ij}]$ be a square matrix of order n such that $|A| = |a_{ij}|$.

Then the minor of an element of the matrix A is defined as the determinant obtained by deleting the row and the column in which the element lies. Minors are required for calculating matrix cofactors.

Consider a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Thus, the minors of a_1, b_1 and c_1 are respectively, $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

Similarly, the minors of a_2, b_2 and c_2 are respectively, $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

Similarly, the minors of a_3, b_3 and c_3 are respectively, $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

5.4.5.2 Cofactor

The cofactor of any element of ith row and jth column is

$$\text{Cofactor} = (-1)^{i+j} \text{ minor}$$

Cofactors are useful for computing both the determinants and inverse of square matrices.

SOME SOLVED EXAMPLES

Example 5.18. *Find all the minors and cofactors of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 7 & 0 & -1 \end{bmatrix}$.

Solution. Given $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 7 & 0 & -1 \end{bmatrix}$$

Here,
$$a_{11} = 1, \quad a_{12} = 2, \quad a_{13} = 3$$
$$a_{21} = 4, \quad a_{22} = 3, \quad a_{23} = 2$$
$$a_{31} = 7, \quad a_{32} = 0, \quad a_{33} = -1$$

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = -3$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 4 & 2 \\ 7 & -1 \end{vmatrix} = -4 - 14 = -18$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 4 & 3 \\ 7 & 0 \end{vmatrix} = 0 - 21 = -21$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2 - 0 = -2$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix} = -1 - 21 = -22$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 0 \end{vmatrix} = 0 - 14 = -14$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-3) = -3$$
$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = (-1)^3 (-18) = 18$$
$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = (-1)^4 (-21) = -21$$
$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$$
$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = (-1)^4 (-22) = -22$$
$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = (-1)^5 (-14) = 14$$
$$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = (-1)^4 (-5) = -5$$
$$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$$
$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = (-1)^6 (-5) = -5$$

5.4.6 Adjoint of a Square Matrix

Adjoint of a square matrix A is obtained by replacing each element of A by its cofactor in $|A|$ and then taking the transpose of the matrix so obtained.

Let the determinant of a square matrix A be $|A|$.

Thus, if $$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \text{ then}$$

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The matrix formed by the cofactors of the elements in $|A|$ is given by,

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

where, $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = (b_2c_3 - c_2b_3)$

$$A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -(b_1c_3 - c_1b_3)$$

$$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)$$

$$B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -(a_2c_3 - c_2a_3)$$

$$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = (a_1c_3 - c_1a_3)$$

$$B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -(a_1c_2 - c_1a_2)$$

$$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - b_2a_3$$

$$C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -(a_1b_3 - b_1a_3)$$

$$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = (a_1b_2 - b_1a_2)$$

Then, the transpose of the matrix of cofactors is $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$ is called the adjoint of the matrix A and is denoted as adj. A.

5.4.6.1 Property of Adjoint

The product of a matrix A and its adjoint is equal to the unit matrix multiplied by the determinant of A.

Symbolically, if A is a square matrix, then

$$\text{adj.}\,(A)\,.\,A = A\,(\text{adj.}\,A) = |\,A\,|\,.\,I$$

where I is a unit matrix.

SOME SOLVED EXAMPLES

Example 5.19. *Find the adjoint of the given matrix* $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

Solution. Given $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

Here, $a_{11} = 2, a_{12} = 3, a_{21} = 3, a_{22} = 5$

Cofactors of a_{11}, a_{12}, a_{21} and a_{22} is given by

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1}(5) = 5$$
$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2}(3) = -3$$
$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1}(3) = -3$$
$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2}(2) = 2$$

Thus, $$\text{adj. }(A) = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \quad \textbf{Answer}$$

Example 5.20. *Find the adjoint of the given matrix* $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

Solution. Given $$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Here, $$a_{11} = 1, a_{12} = 2, a_{13} = 4$$
$$a_{21} = 2, a_{22} = 3, a_{23} = 2$$
$$a_{31} = 3, a_{32} = 3, a_{33} = 4$$

Cofactors of $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$ and a_{33} are calculate as discussed earlier.

Then, we have, $$A_{11} = 6, \quad A_{12} = -2, \quad A_{13} = -3$$
$$A_{21} = 4, \quad A_{22} = -8, \quad A_{23} = 3$$
$$A_{31} = -8, \quad A_{32} = 6, \quad A_{33} = -1$$

Thus, $$\text{adj. }(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$\Rightarrow$ $$\text{adj. }(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$\Rightarrow$ $$\text{adj. }(A) = \begin{bmatrix} 6 & 4 & -8 \\ -2 & -8 & 6 \\ -3 & 3 & -1 \end{bmatrix} \quad \textbf{Answer}$$

5.4.6.2 Inverse of a Matrix

If A and B are two square matrices of the same order, such that $AB = BA = I$, then B is called the inverse of A *i.e.* $B = A^{-1}$ and 'A' is the inverse of B.

Remarks: 1. Condition for a square matrix 'A' to possess an inverse is that matrix 'A' should be non-singular *i.e.* $|A| \neq 0$.

2. Any square matrix which possess an inverse is called an invertible matrix.

3. If any square matrix possess inverse, then it is always unique.

To find the inverse of a matrix 'A' with the help of its adjoint matrix, we have

$$A^{-1} = \frac{\text{adj.}(A)}{|A|} = \frac{1}{|A|}(\text{Adj. } A)$$

(Students can find this result with the help of property of adjoint discussed earlier.)

SOME SOLVED EXAMPLES

Example 5.21. *Find the inverse of the matrix* $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

Solution. Given $\quad A = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \neq 0$

$\Rightarrow$ As $|A| \neq 0$, so 'A' is non-singular and hence A^{-1} exists.

So, we need to find cofactors.

Thus,

$$A_{11} = \text{cofactor of } a_{11} = 5$$
$$A_{12} = \text{cofactor of } a_{12} = -3$$
$$A_{21} = \text{cofactor of } a_{31} = -3$$
$$A_{22} = \text{cofactor of } a_{22} = 2$$

$\Rightarrow$ $$\text{Adj. }(A) = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

Now, $$A^{-1} = \frac{\text{Adj. }(A)}{|A|} = \frac{1}{1}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$\Rightarrow$ $$A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$ **Answer**

Example 5.22. *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ *and also verify that* $AA^{-1} = A^{-1}A = I$.

Solution. Given $$\text{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

So we have, $$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{vmatrix} = 1 \neq 0$$

Thus $|A| \neq 0$ means 'A' is non-singular matrix and hence A^{-1} exists,

To find A^{-1}, we have to find Adjoint of A, for this we will find cofactors of A.

We have
$$A_{11} = -9, \quad A_{12} = -8, \quad A_{13} = -2$$
$$A_{21} = 8, \quad A_{22} = 7, \quad A_{23} = 2$$
$$A_{31} = -5, \quad A_{32} = -4, \quad A_{33} = -1$$

$\therefore$
$$\text{Adj.}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$
$$= \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

Hence,
$$A^{-1} = \frac{1}{|A|}\text{Adj.}(A)$$
$$= \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

For verification, $AA^{-1} = A^{-1}A = I$ (Students can try by themselves.)

EXERCISE 5.4

Questions Based on Minors, Cofactors, Adjoint and Inverse using Adjoint for a given Matrix

1. Find all the minors and cofactors of each element for the given determinant. Also solve the given determinant $A = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$.

2. Find the adjoint of the given matrices:

a. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

3. Find the adjoint of the given matrix $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

4. Find the inverse of the following matrices:

a. $\begin{bmatrix} 1 & 2 & 3 \\ -3 & 5 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix}$

5. Find the inverse of the given matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ -2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

6. Given $D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$ where none d_1, d_2, d_3 and d_4 are zero. Find D^{-1}.

Answers

1. $M_{11} = (ab^2 - ac^2)$, $M_{12} = (ab - ac)$, $M_{13} = (c - b)$
$M_{21} = a^2b - bc^2$, $M_{22} = (ab - bc)$, $M_{23} = (c - a)$
$M_{31} = (ca^2 - cb^2$, $M_{32} = (ca - bc)$, $M_{33} = (b - a)$
$A_{11} = (ab^2 - ac^2)$, $A_{12} = (ac - ab)$, $A_{13} = (c - b)$
$A_{21} = (bc^2 - a^2b)$, $A_{22} = (ab - bc)$, $A_{23} = (a - c)$
$A_{31} = (ca^2 - cb^2)$, $A_{32} = (bc - ca)$, $A_{33} = (b - a)$, $|A| = (a - b)(b - c)(c - a)$

2. a. $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ b. $\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

4. a. $\frac{1}{2}\begin{bmatrix} 5 & 1 & -15 \\ 3 & 1 & -9 \\ -3 & -1 & 11 \end{bmatrix}$ b. $\frac{1}{11}\begin{bmatrix} 7 & -5 \\ -2 & 3 \end{bmatrix}$

5. $\frac{1}{5}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -20 \\ 2 & 3 & -15 \end{bmatrix}$ **6.** $\begin{bmatrix} 1/d_1 & 0 & 0 & 0 \\ 0 & 1/d_2 & 0 & 0 \\ 0 & 0 & 1/d_3 & 0 \\ 0 & 0 & 0 & 1/d_4 \end{bmatrix}$

INTERESTING FACTS

- The structure of various buildings can be changed or designed with the help of matrices. We can take an example of "***Burj Khalifa***". The design which is not so common was made by using matrices.
- Some specially designed functions, such as the ***Iterated Function Systems***, are really fun to draw and are computed with the use of matrices.
- In the domain of IT and Information Security (especially encryption), many IT companies also use these matrices as data structures to perform search queries, track user information, and manage databases.
- It can also be used at many places, such as the matrix rotating while playing the ***car racing game***; building a cluster of networks over the ***Face book, Twitter, Instagram***; or even while trading in the ***Wall Street***.

VIDEO REFERENCES

Basic Matrix Concepts

Introduction to Matrix Algebra - I

Matrix Analysis with Applications

Determinant of a Matrix

Elementary Row Operations

APPLICATIONS TO REAL LIFE

- Various Graphic software such as Adobe Photoshop on your personal computer uses matrices to process linear transformations to render images.
- A square matrix can represent a linear transformation of a geometric object, for example, a matrix reflects an object in x or y-axis in a cartesian x-y plane.
- It also has its application in the domain of gaming industry and image processing domain, where reflections in ponds, rivers, and other upside-down images are seen.
- Matrices also play an important role in computer graphics, like when people want to apply any of the desired matrix transformations on any object, for example cartoon characters.
- They also play significant role in plotting surveys, representation of real-world data such as the population of people, infant mortality rate can be done through them.
- Even in economics, constructing the predictive model of dependent variables, analysing the shares, studying the trends of business can be done through matrices.
- In physics, while calculating the battery power outputs, solving Kirchhoff's Law, and in the field of quantum physics, matrix does play an important part.
- Also in geology, they play a crucial part while making seismic surveys.
- In robotics, the robotic movements are defined on the basis of matrices.

5.5 GAUSS ELIMINATION METHOD FOR FINDING THE INVERSE OF A MATRIX

Let A be a non-singular square matrix of order three. Then the inverse of A is a matrix X which satisfy the equation $AX = I$, where I is the unit matrix of order three. Now, we have to find the elements of the inverse matrix X.

Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

The equation becomes $AX = I$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix is equivalent to three equations, which are equivalent to three system of equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{...(1)}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{...(2)}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{...(3)}$$

The systems (1), (2) and (3) of Eqs. (1)–(3) can be solved by Gauss-elimination procedure. The solution set of each system of Eqs. (1), (2) and (3) will be the corresponding column of the inverse matrix X.

Since the coefficient matrix is same in all the Eqs. (1), (2) and (3), all can be simultaneously solved by forming a definite system

$$[A/I] = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

SOME SOLVED EXAMPLES

Example 5.23. *By Gauss elimination Method, find the inverse of* $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{pmatrix}$.

Solution. The augmented system $[A/I]$ is

$$[A/I] \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right)$$

Since the element $a_{11} = 0$, we will interchange the first and second row, the reduced system is

$$[A/I] \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right)$$

we get $$R_3 \to R_3 + (-3)R_1 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & 0 & -3 & 1 \end{array}\right)$$

$$R_3 \to R_3 + 7R_2 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right)$$

Thus $$\left.\begin{aligned} x_{11} + 2x_{21} &= 0 \\ x_{21} + x_{31} &= 1 \\ 3x_{31} &= 7 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{31} &= \frac{7}{3} \\ x_{21} &= -\frac{4}{3} \\ x_{11} &= \frac{8}{3} \end{aligned}$$

$$\left.\begin{aligned} x_{12} + 2x_{22} &= 1 \\ x_{22} + x_{32} &= 0 \\ 3x_{32} &= -3 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{32} &= -1 \\ x_{22} &= 1 \\ x_{12} &= -1 \end{aligned}$$

$$\left.\begin{aligned} x_{13} + 2x_{23} &= 0 \\ x_{23} + x_{33} &= 0 \\ 3x_{33} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{33} &= \frac{1}{3} \\ x_{23} &= -\frac{1}{3} \\ x_{13} &= \frac{2}{3} \end{aligned}$$

Hence $$A^{-1} = \begin{pmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ -\frac{4}{3} & 1 & -\frac{1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{pmatrix}$$

Example 5.24. *Find by Gauss elimination method, the inverse of* $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$.

Solution. $$[A/I] = \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \to R_2 + 5R_1, R_3 \to R_3 - \left(\frac{5}{3}\right)R_1 \sim \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & \frac{-5}{3} & 0 & 1 \end{array}\right)$$

$$R_3 \to R_3 + \left(\frac{1}{3}\right)R_2 \sim \left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right)$$

Now the system is equivalent to three systems.

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & \frac{1}{3} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

and

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \end{array}\right)$$

$$\left.\begin{aligned} 3x_{11} - x_{21} + x_{31} &= 1 \\ x_{21} &= 5 \\ \frac{1}{3}x_{31} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{31} &= 0 \\ x_{21} &= 5 \\ x_{11} &= 2 \end{aligned}$$

$$\left.\begin{aligned} 3x_{12} - x_{22} + x_{32} &= 0 \\ x_{22} &= 1 \\ \frac{1}{3}x_{32} &= \frac{1}{3} \end{aligned}\right\} \Rightarrow \begin{aligned} x_{32} &= 1 \\ x_{22} &= 1 \\ x_{12} &= 0 \end{aligned}$$

$$\left.\begin{aligned} 3x_{13} - x_{23} + x_{33} &= 0 \\ x_{23} &= 0 \\ \frac{1}{3}x_{33} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{33} &= 3 \\ x_{23} &= 0 \\ x_{13} &= -1 \end{aligned}$$

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

EXERCISE 5.5

1. Find the inverse of the following matrices by Gauss elimination method:

i. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ ii. $\begin{pmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{pmatrix}$ iii. $\begin{pmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{pmatrix}$

Answers

1. i. $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ ii. $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$ iii. $\frac{1}{8}\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

5.6 GAUSS-JORDAN METHOD FOR FINDING THE INVERSE OF A MATRIX

Let A be square matrix of order three and $|A| \neq 0$. Then the inverse of A is a matrix X which satisfies the equation $AX = I$, where I is the unit matrix of order three. Now, we have to find the elements of the inverse matrix X.

Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Then,
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This equation is equivalent to three equations, which are equivalent to three systems of equations. Solve each system by Gauss-Jordan method. The solution set of each system will be corresponding column of the inverse matrix. Here also we can solve all the systems simultaneously by forming the augmented system.

$$[A/I] = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

SOME SOLVED EXAMPLES

Example 5.25. *Find the inverse of* $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ *by Gauss-Jordan method.*

Solution. The augmented system $[A/I]$ is

$$[A/I] = \left(\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 \to \frac{1}{3}R_1 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \to R_2 - 2R_1 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \to (-1)R_2 \sim \left(\begin{array}{ccc|ccc} 1 & -1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 \to R_1 + R_2 \text{ and } R_3 \to R_3 + R_2 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 1 \end{array}\right)$$

$$R_3 \to (-3)\, R_3 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

$$R_2 \to R_2 + \frac{4}{3}R_3 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right)$$

Inverse of $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

EXERCISE 5.6

1. Find the inverse of the following matrices by Gauss-Jordan method:

i. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ ii. $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ iii. $\begin{pmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{pmatrix}$ iv. $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$

v. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{pmatrix}$

Answers

1. i. $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ ii. $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

iii. $\frac{1}{8}\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$ iv. $\begin{pmatrix} -\frac{4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & -\frac{8}{3} \\ \frac{7}{3} & -3 & -\frac{10}{3} \end{pmatrix}$

v. $\begin{pmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ -\frac{4}{3} & 1 & -\frac{1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{pmatrix}$

INTERESTING FACTS

- It has a unique application in the domain of environmental sciences, like, in determining average caloric value of specific fishes, growth of forest and its types, etc. For example, if there are 5 different types of trees in a forest, we can create a linear equation to measure their age. Similarly, we can create linear equations to find the intake of carbohydrates of marine life based on their diet, and there are many such examples.
- In real world, we can apply this in ***Traffic Control Management*** to tackle the issue of Traffic jams using ***Gauss Jordan method***, involving the technique of finding inverse of a matrix, by forming ***neutrosophic linear equations*** (which revolves around unrealistic dataset and represents determinate and/or indeterminate information)and applying MATLAB programming.

APPLICATIONS TO REAL LIFE

- Finding the values of unknown quantities; let it be age, cost, or any other thing.
- At airports, where high-end computers are used to calculate and encode information about flights, passengers, etc.

- In circuit analysis, Gauss Jordan process is used on mesh-connected processors.
- It is used in scheduling algorithms.
- Useful in Fingerprint Image Enhancement, which involves the application of Gaussian Elimination method.

5.7 THEOREMS BASED ON SYMMETRIC AND SKEW-SYMMETRIC (Anti-symmetric) MATRICES

We have already discussed the definitions of symmetric and skew-symmetric matrices.

Here we will discuss some theorems based on them:

Theorem 1: *The necessary and sufficient condition for a matrix A to be symmetric is that $A = A'$.*

Proof: The condition is necessary.

Let $A = [a_{ij}]$ be a n-rowed square symmetric matrix.

This means $a_{ij} = a_{ji}$ and A' *i.e.* the transpose of A is also n-rowed square matrix.

Now $(i, j)^{th}$ element of A'

$$= (j, i)^{th} \text{ element of } A$$

$\because$ A is symmetric $\Rightarrow$ $a_{ij} = a_{ji} \ \forall \ i, j$

$$= (i, j)^{th} \text{ element of } A$$

$\therefore$ $A = A'$

The condition is sufficient

Here $A = A'$

To prove: A is symmetric.

If $A = A'$, A must be n-rowed square matrix.

Also $(i, j)^{\text{th}}$ element of A $= (i, j)^{\text{th}}$ element of A' $[\because \ A = A']$

$$= (j, i)^{\text{th}} \text{ element of } A$$

$\therefore$ A is symmetric.

Theorem 2: *The necessary and sufficient condition for a matrix A to be skew-symmetric is that $A' = -A$.*

Proof: Let $A = [a_{ij}]$ be a n-rowed square skew-symmetric matrix. Then

$$a_{ij} = -a_{ji}.$$

Since, 'A' is n-rowed square matrix, A', $-A$ are also n-rowed square matrices.

Now $(i, j)^{\text{th}}$ element of $A' = (j, i)^{th}$ element of $(-A)$

Since 'A' is skew-symmetric $\Rightarrow$ $a_{ij} = -a_{ji} \ \forall i, j$

$$= (i, j)^{th} \text{ element of } (-A)$$

$\Rightarrow$ $A' = -A$

Conversely; If $A' = -A$, then A must be a square matrix

Also $(i, j)^{th}$ element A = the negative of the $(i, j)^{th}$ element of A' $[\because \ -A' = A]$

$$= \text{the negative of } (j, i)^{th} \text{ element of } A.$$

Hence A is a skew-symmetric matrix.

Theorem 3: *If A is a skew-symmetric and X is a column matrix, then show that $X' AX$ is a null matrix.*

Proof: Since A is a skew-symmetric matrix, then $A' = -A$

Let A be a square matrix of order n and X be a column matrix of order $n \times 1$. Now X' is a row matrix of order $1 \times n$. Hence $X'\,AX$ is a matrix of order 1×1.

Let $$X'AX = B \qquad ...(1)$$

Since B is of order 1×1, then $B' = B$, and hence B is symmetric.

Now consider $$(X'AX)' = B'$$

$\therefore$ $$X'\,A'(X')' = B' \quad \Rightarrow \quad X'A'X'' = B'$$

But $$X'' = X, A' = -A \text{ and } B' = B$$

$\therefore$ We have, $$X'\,(-A)\,X = B$$

$\Rightarrow$ $$-(X'\,AX) = B$$

$\Rightarrow$ $$-B = B \qquad \text{(From (1))}$$

$\Rightarrow$ $$2B = 0$$

$\therefore$ $$B = 0$$

$\Rightarrow$ $X'AX$ is a null matrix.

Theorem 4: *Show that every square matrix can be uniquely expressed as the sum of two matrices, one symmetric and other anti-symmetric.*

Proof: Let A be a given square matrix then 'A' can be written as

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$
$$= P + Q \text{ (say)}$$

where $$P = \frac{1}{2}(A + A'), Q = \frac{1}{2}(A - A')$$

Now we will show that P is a symmetric matrix and Q is a skew-symmetric matrix.

For this, let $$P' = \frac{1}{2}(A + A')' = \frac{1}{2}[(A') + (A')']$$
$$= \frac{1}{2}[A' + A]$$
$$= \frac{1}{2}[A + A'] = P$$

$\therefore$ P is symmetric.

Also $$Q' = \frac{1}{2}[A - A']'$$
$$= \frac{1}{2}[A' - (A')']$$
$$= \frac{1}{2}(A' - A) = -\frac{1}{2}[A - A']$$
$$= -Q$$

$\therefore$ Q is skew-symmetric

Thus, we expressed 'A' as the sum of symmetric and skew-symmetric matrix.

To prove uniqueness:

Let $A = R + S$, where R is symmetric and S is skew-symmetric be another representation of A.

Now, $A' = (R + S)' = R' + S'$

$= R - S$ $\{\because \quad R' = R, S' = -S]$

$\therefore \quad \frac{1}{2}(A + A') = \frac{1}{2}[(R + S) + (R - S)] = R$

$\therefore \quad R = P$

and $\frac{1}{2}(A - A') = \frac{1}{2}[(R + S) - (R - S)] = S$

$\therefore \quad S = Q$

Hence the representation $A = P + Q$ is unique.

SOME SOLVED EXAMPLES

Example 5.26. *If* $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$, *then represent it as $A = B + C$, where B is symmetric and C is skew-symmetric.*

Solution. We have, $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$

On transposing, we get $A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

On adding A and A', we get

$$A + A' = \begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix} \quad \text{...(1)}$$

On subtracting A' from A, we get

$$A - A' = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix} \quad \text{...(2)}$$

On adding (1) and (2), we have

$$2A = \begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$$

or

$$A = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$= \text{Symmetric} + \text{skew-symmetric}$

Example 5.27. *Express* $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$ *as a sum of symmetric and skew-symmetric matrix.*

Solution. Let,
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$$

On transposing, we have $A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 0 & 1 & 3 \end{bmatrix}$

Adding A and A',
$$A + A' = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 14 & 10 \\ 5 & 10 & 6 \end{bmatrix} \qquad \text{...(1)}$$

On subtracting, A' from A, we have
$$A - A' = \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & -8 \\ 5 & 8 & 0 \end{bmatrix} \qquad \text{...(2)}$$

Adding (1) and (2), we get
$$2A = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 14 & 10 \\ 5 & 10 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & -8 \\ 5 & 8 & 0 \end{bmatrix}$$

or
$$A = \begin{bmatrix} 1 & 5/2 & 5/2 \\ 5/2 & 7 & 5 \\ 5/2 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & -4 \\ 5/2 & 4 & 0 \end{bmatrix}$$
$$= \text{Symmetric} + \text{skew-symmetric}$$

Example 5.28. *If 'A' is a skew-symmetric matrix of odd order, or even order, then by taking an example prove that the determinant of 'A' is always '0' or real number respectively:*

Solution. (1) Let, $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$, '$A$' is odd order and skew-symmetric matrix, then
$$|A| = 0(0+16) - 2(0+12) + 3(8+0)$$
$$= 0 - 24 + 24 = 0$$

(2) If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *i.e.*, 'A' is even order matrix and skew-symmetric, then
$$|A| = 0 + 1 = 1 \text{ } i.e., \text{ a real number.}$$

5.8 ORTHOGONAL MATRIX

A real square matrix A is called orthogonal if $AA' = A'A = I$.

5.8.1 Properties of Orthogonal Matrix

a. *If A is an Orthogonal Matrix, then $|A| = \pm 1$.*

Proof: Since determinant remains unchanged by interchanging of rows and columns, $|A'| = |A|$.

Further by definition, if A is orthogonal, then $AA' = I$

$$\therefore \quad |AA'| = |I|$$
$$\Rightarrow \quad |A||A'| = |I|$$
$$\Rightarrow \quad |A|^2 = 1$$
$$\Rightarrow \quad |A| = \pm 1$$

> **Remark:** If A is an orthogonal square matrix of order n, since $|A| = \pm 1$, then rank of $A = n$.

b. *If A is an orthogonal matrix, then A^{-1} exists and is equal to A'.*

Proof: We know that if A and B are two matrices such that $AB = BA = I$, then B is called the inverse of A and the necessary and the sufficient condition for A to have inverse is $|A| \neq 0$

As seen in (a) if A is orthogonal, then $|A| = \pm 1 \neq 0$, $\quad \therefore \quad A^{-1}$ exists.

Further $\quad AA' = I, \quad \therefore \quad A^{-1}(AA') = A^{-1} \,.\, \mathrm{I}$

$$\therefore \quad (A^{-1}A) \,.\, A' = A^{-1}$$
$$\therefore \quad IA' = A^{-1}$$
$$\Rightarrow \quad A' = A^{-1}$$

c. *If A and B are two orthogonal square matrices of order n, then AB and BA are also orthogonal.*

Proof: Since A and B are square matrices of order n. AB and BA are defined and are square matrices of order n.

Since A, B are orthogonal,

$$|A| \neq 0, |B| \neq 0 \ (= \pm 1) \text{ and } A^{-1}, B^{-1} \text{ exists}$$

Further, $\quad |AB| = |A||B| \neq 0$

$\therefore \quad (AB)^{-1}$ exists.

Now $\quad (AB)' = B'A'$

$$\text{Further} \quad (AB)'(AB) = B'A'AB$$
$$= B'(A'A)B$$
$$= B'IB = B'B = I$$

Hence $(AB)'$ is the inverse of AB

$\therefore \quad AB$ is orthogonal.

Note: If A is an orthogonal matrix, then $|A| = \pm 1$

If $|A| = 1$, then A is called proper orthogonal matrix.

> **Remark:** Numerical analysis takes advantage of many of the properties of **orthogonal matrices** for numerical linear algebra. For **example**, it is often desirable to compute an **orthonormal** basis for an inner product space, or an **orthogonal** change of bases; both take the form of **orthogonal matrices**.

SOME SOLVED EXAMPLES

Example 5.29. *Verify that* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ *is orthogonal.*

Solution. Here
$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$\therefore$
$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

and
$$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence 'A' is an orthogonal matrix.

Example 5.30. *Determine the values of* α, β, γ *when* $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ *is orthogonal.*

Solution. Let
$$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$

On transposing A, we have
$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

If A is orthogonal, then $AA' = I$

$\therefore$
$$AA' = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0+4\beta^2+\gamma^2 & 0+2\beta^2-\gamma^2 & 0-2\beta^2+\gamma^2 \\ 0+2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ 0-2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad 4\beta^2 + \gamma^2 = 1 \qquad \text{...(1)}$$

and $$-2\beta^2 + \gamma^2 = 0 \quad \Rightarrow \quad \beta^2 = \frac{\gamma^2}{2} \qquad \text{...(2)}$$

$$(1) \Rightarrow \qquad 4 \cdot \frac{\gamma^2}{2} + \gamma^2 = 1 \qquad \left(\text{Putting } \beta^2 = \frac{\gamma^2}{2}\right)$$

or $$2\gamma^2 + \gamma^2 = 1 \quad \Rightarrow \quad 3\gamma^2 = 1 \quad \Rightarrow \quad \gamma = \pm \frac{1}{\sqrt{3}}$$

Also $$\beta^2 = \frac{1}{6} \quad (\text{from (2)}) \quad \Rightarrow \quad \beta = \pm \frac{1}{\sqrt{6}}$$

Also, we have $$\alpha^2 = 1 - \beta^2 - \gamma^2 \qquad [\because \quad \alpha^2 + \beta^2 + \gamma^2 = 1]$$

$$= 1 - \frac{1}{6} - \frac{1}{3}$$

$$\alpha^2 = \frac{6-1-2}{6} = \frac{1}{2}$$

$$\Rightarrow \qquad \alpha = \pm \frac{1}{\sqrt{2}}$$

Example 5.31. *If* $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, *prove that* $A^{-1} = A'$, A' *being the transpose of* A.

Solution. We have, $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, and $A' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$, then

$$AA' = \frac{1}{81}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad AA' = I \quad \Rightarrow \quad A' = A^{-1}$$

Example 5.32. *Is the matrix* $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ *orthogonal?*

Solution. Transpose of A is $A' = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

Consider
$$A'A = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 4+16+9 & -6+12-3 & 2+4-27 \\ -6+12-3 & 9+9+1 & -3+3+9 \\ 2+4-27 & -3+3+9 & 1+1+81 \end{bmatrix}$$
$$= \begin{bmatrix} 29 & 3 & -21 \\ 3 & 19 & 9 \\ -21 & 9 & 83 \end{bmatrix} \neq I$$

So, matrix A is not orthogonal.

Example 5.33. *Is the matrix* $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ *orthogonal?*

Solution. The transpose matrix $A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

So,
$$A'A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta+0+\sin^2\theta & 0+0+0 & \cos\theta\sin\theta-\cos\theta\sin\theta \\ 0+0+0 & 0+1+0 & 0+0+0 \\ \cos\theta\sin\theta-\cos\theta\sin\theta & 0+0+0 & \sin^2\theta+0+\cos^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the given matrix is orthogonal.

EXERCISE 5.7

1. If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$, then find $(AB)'$. Hence verify also $(AB)' = B'A'$.

2. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that AA' and $A'A$ are both symmetric matrices.

3. If A and B are symmetric matrices, then show that $AB - BA$ is a skew-symmetric matrix.

4. Express the matrix 'A' given below as the sum of a symmetric and a skew-symmetric matrix.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$$

5. Prove that the following matrix is orthogonal and hence find A^{-1}.

$$A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$ is orthogonal, find a, b and c.

7. Check the given matrix is orthogonal or not?

$$A = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

8. Is the following matrix orthogonal?

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

9. Prove that the following matrices are orthogonal and hence find A^{-1} also.

i. $\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ ii. $\frac{1}{\sqrt{6}}\begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ 2 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$ iii. $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. If $3A = \begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and if A is orthogonal, find a, b, c.

11. Express the matrix $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

12. Check whether the following matrices are proper orthogonal or improper orthogonal:

i. $\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$ ii. $\begin{bmatrix} 12/13 & 5/13 \\ -5/13 & 12/13 \end{bmatrix}$ iii. $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Answers

1. $\begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$

4. Symmetric matrix $= \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 5 & 9/2 \\ 3/2 & 9/2 & 3 \end{bmatrix}$; Skew-symmetric matrix $= \begin{bmatrix} 0 & 2 & 5/2 \\ -2 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix}$

5. A' is the inverse of A **6.** $a = \pm\frac{2}{3}, b = \mp\frac{2}{3}, c = \pm\frac{1}{3}$

7. No **8.** No

9. i. $\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$ ii. $\frac{1}{\sqrt{6}}\begin{bmatrix} 0 & 2 & \sqrt{2} \\ \sqrt{3} & 1 & -\sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \end{bmatrix}$ iii. $\frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. $a = 2, b = 2, c = 1$

11. Symmetric Matrix $= \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}$; skew-symmetric Matrix $= \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}$

12. i. Improper ii. Proper iii. Not Orthogonal

5.9 VECTORS

Any vector in two dimension is represented by $X = (x_1, x_2)$ and in three dimension as $X = (x_1, x_2, x_3)$. Similarly, we may represent vector in 4th dimensional space as $X = (x_1, x_2, x_3, x_4)$ and so on.

In n dimensional space vector X which is represented by such numbers (called ordered n-tuple numbers) *i.e.*, $X = (x_1, x_2, x_3, \ldots x_n)$ is called an n-vector in which x_i is fixed for $i = 1, 2, 3, \ldots, n$.

Numbers $x_1, x_2, x_3, \ldots x_n$ with in brackets are called components of vector $X = (x_1, x_2, x_3, \ldots, x_n)$.

A vector is written either in rows or columns such as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ or } X = [x_1, x_2, x_3]$$

A null vector consists of all elements as zeroes and is represented by O, *i.e.*, $O = (0, 0, 0, \ldots, 0)$

If we take two vectors in ordered n-tuples as

$$X = (x_1, x_2, x_3, ..., x_n) \text{ and } Y = (y_1, y_2, y_3, ..., y_n)$$

Then above two vectors are defined as equal if each element of X is equal to corresponding element of Y.

i.e., $\quad x_i = y_i$ for all $i = 1, 2, ..., n$ then $X = Y$

and sum of the above two vectors is defined as follows:

$$X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, ..., x_n + y_n)$$

and scalar multiple of vector X is defined as

$$kX = (kx_1, kx_2, kx_3, ..., kx_n)$$

such that $\quad mX + nY = (mx_1 + ny_1, mx_2 + ny_2, mx_3 + ny_3, ..., mx_n + ny_n)$

5.9.1 Linear Dependence and Independence of Vectors

Let $X_1, X_2, X_3, ... X_r$ be n-tupled vectors r in numbers, if corresponding to above r vectors, we could find r scalars, K_i $(i = 1, 2, 3, 4, ..., r)$, in which all are not zeroes (a few may be zero) such that

$$K_1X_1 + K_2X_2 + K_3X_3 + ... + K_rX_r = 0 \qquad ...(1)$$

then these vectors X_i $(i = 1, 2, ..., r)$ are said to be dependent.

If (1) is satisfied such that $\quad K_1 = K_2 = ... = K_r = 0$, then

The vector X_i $(i = 1, 2, ..., r)$ called independent vectors.

SOME SOLVED EXAMPLES

Example 5.34. *Are the following vectors linearly dependent? If so, find the relation between them, $x_1 = (3, 2, 7)$, $x_2 = (2, 4, 1)$ and $x_3 = (1, -2, 6)$.*

Solution. Let λ_1, λ_2 and λ_3 be the scalars such that

$$\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 = 0 \qquad ...(1)$$

$$\Rightarrow \quad \lambda_1(3, 2, 7) + \lambda_2(2, 4, 1) + \lambda_3(1, -2, 6) = 0$$

$$\Rightarrow \quad 3\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \qquad ...(A)$$

$$2\lambda_1 + 4\lambda_2 - 2\lambda_3 = 0 \qquad ...(B)$$

$$7\lambda_1 + \lambda_2 + 6\lambda_3 = 0 \qquad ...(C)$$

Eliminating λ_3 from (A) and (B) we get $\lambda_1 + \lambda_2 = 0$ and also putting $\lambda_1 + \lambda_2 = 0$ in (C), we get

$$\lambda_2 - \lambda_3 = 0$$

$$\Rightarrow \quad \lambda_2 = \lambda_3 = -\lambda_1$$

so, equation (1) becomes $\lambda_1(x_1 - x_2 - x_3) = 0$, but $\lambda_1 \neq 0$

so, $\quad x_1 - x_2 - x_3 = 0 \quad \Rightarrow \quad x_1 = x_2 + x_3$, *i.e.*, vectors are linearly dependent.

Example 5.35. *Are the following vectors linearly dependent? If so, find the relation between them, $x_1 = (1, 1, 2, 3)$, $x_2 = (1, 2, 3, 4)$ and $x_3 = (2, 3, 4, 9)$.*

Solution. Let λ_1, λ_2 and λ_3 be scalars

such that $\quad \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 = 0 \qquad ...(1)$

$$\Rightarrow \quad \lambda_1(1, 1, 2, 3) + \lambda_2(1, 2, 3, 4) + \lambda_3(2, 3, 4, 9) = 0$$

$$\Rightarrow \quad \lambda_1 + \lambda_2 + 2\lambda_3 = 0 \qquad ...(A)$$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \quad \text{...(B)}$$
$$2\lambda_1 + 3\lambda_2 + 4\lambda_3 = 0 \quad \text{...(C)}$$
$$3\lambda_1 + 4\lambda_2 + 9\lambda_3 = 0 \quad \text{...(D)}$$

Now from (B) – (A), we get $\lambda_2 + \lambda_3 = 0$ ...(E)

And (C) – (B) provides $\lambda_1 + \lambda_2 + \lambda_3 = 0$ ...(F)

From (E) and (F), $\lambda_1 = 0$

$\therefore$ Putting $\lambda_1 = 0$ and $\lambda_2 = -\lambda_3$ in (B), we get

$$\lambda_2 = \lambda_3 = 0$$

$\Rightarrow$ $\lambda_1 = 0, \lambda_2 = 0$ and $\lambda_3 = 0$

$\Rightarrow$ Vectors x_1, x_2 and x_3 represented by (1) are linearly independent vectors (and non dependent vectors).

Example 5.36. *Examine the following vectors for linear dependence and find the relation if exists.*

$$x_1 = (1, 2, 4), x_2 = (2, -1, 3), x_3 = (0, 1, 2), x_4 = (-3, 7, 2)$$

Solution. Consider the matrix equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0 \quad \text{...(1)}$$

$\Rightarrow$ $\lambda_1 (1, 2, 4) + \lambda_2 (2, -1, 3) + \lambda_3 (0, 1, 2) + \lambda_4 (-3, 7, 2) = 0$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$
$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$
$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

This is the homogeneous system.

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $A\lambda = 0$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$
$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0 \quad \text{...(2)}$$
$$\lambda_3 + \lambda_4 = 0$$

Let $\lambda_4 = t, \lambda_3 = -t$

$$-5\lambda_2 - t + 13t = 0$$

then, from (2),

$$\Rightarrow \qquad \lambda_2 = \frac{12t}{5}$$

Again, we have $\lambda_1 + \dfrac{24t}{5} - 3t = 0$

$$\text{or} \qquad \lambda_1 = \frac{-9t}{5}$$

Hence the given vectors are linearly dependent.

substituting the values of 'λ' in (1)

we get,

$$\frac{-9t}{5} x_1 + \frac{12t}{5} x_2 - t\,x_3 + t\ x_4 = 0$$

$$\text{or} \qquad -\frac{9x_1}{5} + \frac{12x_2}{5} - x_3 + x_4 = 0$$

$$\text{or} \qquad 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

Example 5.37. *Is the system of vectors $x_1 = (2, 2, 1)^T$, $x_2 = (1, 3, 1)^T$, $x_3 = (1, 2, 2)^T$ linearly dependent? Explain.*

Solution. Here $\qquad x_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Consider the matrix equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0 \qquad \text{...(1)}$$

$$\lambda_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$$

$$2\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$2\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$$
$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$

which is a homogeneous equation

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$ $\qquad\qquad$ $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \lambda_3 = 0, \lambda_2 = 0, \lambda_1 = 0$$

Thus, all scalars are zero. Therefore the given system of vectors is linearly independent (not dependent).

5.9.2 Linear transformations

Let $P(x, y)$ be a point in the dimensions with x, y as axes and these axes are rotated to an angle θ. Keeping the origin O fixed such that the new co-ordinate of P is (x', y') with therefore to the new axes Ox' and Oy' respectively, then

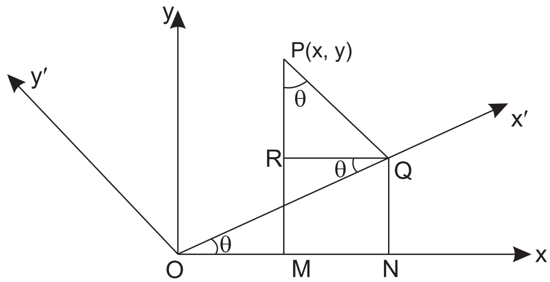


Fig. 5.3

$$OM = x \text{ and } PM = y$$
$$OQ = x' \text{ and } PQ = y'$$

so that $\qquad OM = ON - MN$

$$x = x' \cos\theta - y' \sin\theta \quad ...(1)$$

and $\qquad PM = PR + RM = PR + QN$

$$= y' \cos\theta + x' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta \qquad ...(2)$$

From (1) and (2), we get linear relation in the form

$$x' = a_1 x + b_1 y$$

and
$$y' = a_2 x + b_2 y$$

which can be represented in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow X' = AX \qquad ...(3)$$

This transformation (3) is linear transformation in two dimensions.

Similarly, if we take the relation of the form.

$$x' = a_1 x + b_1 y + c_1 z$$
$$y' = a_2 x + b_2 y + c_2 z$$
$$z' = a_3 x + b_3 y + c_3 z$$

which can be represented in the matrix as

$$X' = AX \text{ where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, X' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \text{ and } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then $X' = AX$ represents a linear transformation in three dimension.

5.9.3 Orthogonal Transformation

The linear transformation $Y = AX$ is called orthogonal if it transforms $y_1^2 + y_2^2 + \dots y_n^2$ into $x_1^2 + x_2^2 + \dots x_n^2$. The matrix A of this transformation is called orthogonal matrix.

i.e., if $$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } X' = [x_1\ x_2\ \dots\ x_n]$$

so that $$X'X = [x_1\ x_2\ \dots\ x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

$$= x_1^2 + x_2^2 + \dots x_n^2$$

and similarly $$Y'Y = y_1^2 + y_2^2 + \dots y_n^2$$

If we consider the transformation $Y = AX$ to be an orthogonal transformation, then

$$Y'Y = (AX)'\,(AX) = (X'A')\,(AX) = X'A'\,AX$$

$$= X'\,(A'A)X = X'IX = X'IX \text{ is true only.}$$

if $$A'A = I \quad \Rightarrow \quad A' = A^{-1} \text{ since } A^{-1}A = I$$

i.e., for an orthogonal transformation $A' = A^{-1}$. Hence for any square matrix A, A is said to orthogonal

if $$AA' = A'A = I \text{ where } A' = A^{-1}$$

Also since $$|A'| = |A|, A'A = I \quad \Rightarrow \quad |A'| = |I| = 1$$

$\Rightarrow$ $$|A||A| = I \quad \Rightarrow \quad |A^2| = 1 \quad \Rightarrow \quad |A| = \pm 1$$

Let $n \times n$ matrix $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$ and column matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and column matrix } Y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} \text{ such that}$$

$$Y = AX \qquad \dots(1)$$

Then (1) defines, a linear transformation, which transforms any vector X into another vector Y, using the matrix A. Matrix A is said to be the linear operator of transformation from vector X to vector Y because the linearity property of vectors is satisfied.

i.e., if X_1 and X_2 are two vector such that

$$Y_1 = AX_1 \text{ and } Y_2 = AX_2 \quad \Rightarrow \quad aY_1 + bY_2 = A(aX_1 + bX_2)$$

If $|A| \neq 0$, then the transformation matrix A is called non-singular and then the linear transformation is known as regular (or non-singular) and if $|A| = 0$, then the transformation of matrix A is called singular and the linear transformation is known as singular. If the transformation of matrix A is singular (non-singular) then the transformation is accordingly called singular (non-singular). Also for $a\ Y = AX$ (a non-singular transformation), $X = A^{-1}\ Y$ is the inverse transformation.

Further, if a vector $X = (x_1, x_2, x_3)$ is transformed by A to vector $Y = (y_1, y_2, y_3)$ such that $Y = AX$ and by another transformation B vector Y is transformed to vector $Z = (z_1, z_2, z_3)$ so that we have $Z = BY$ then $Z = BY = B(AX) = (BA)Z$.

This transformation BA is called composite transformation.

SOME SOLVED EXAMPLES

Example 5.38. *Show the transformation $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ is regular. Write down the inverse transformation*

Solution. If the transformation be taken as $Y = AX$ such that $X = A^{-1}\ Y$,

where
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$\therefore$ The determinant of A is $|A| = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} = -1 \neq 0$. So that transformed matrix A is non-singular and the transformation is regular.

Now, if
$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ then to find } A^{-1} \text{ we find}$$

Co-factors of A corresponding to $a_1, b_1, c_1, a_2, b_2, c_2$ etc., which are

$$A_1 = -2, B_1 = 4, C_1 = -1, A_2 = 2, B_2 = -5, C_2 = 1,$$
$$A_3 = 1, B_2 = -3 \text{ and } C_3 = 1.$$

Now, we know that

$$A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{(-1)} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

$\therefore$ Inverse transformation

$\therefore$
$$X = A^{-1}\ Y \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From which, we get, $x_1 = 2y_1 - 2y_2 - y_3$

$x_2 = -4y_1 + 5y_2 + 3y_3$

and $x_3 = y_1 - y_2 - y_3$

which is the inverse transformation.

Example 5.39. *Show that the transformation*

$$y_1 = x_1 + x_2 + 2x_3$$
$$y_2 = 2x_1 + x_3$$
$$y_3 = -x_1 + x_2 + 3x_2$$

is non-singular. Find the inverse transformation.

Solution. The given system can be written as follows:

$$Y = AX \text{ where } Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Now, $|A| = -4 \neq 0$, the matrix is non-singular and the given transformation is non-singular. The inverse of A can be found by using any suitable method

$$A^{-1} = \begin{bmatrix} 1/4 & 1/4 & -1/4 \\ 7/4 & -5/4 & -3/4 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$

Inverse transformation is $X = A^{-1}Y$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & -1/4 \\ 7/4 & -5/4 & -3/4 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is

$$x_1 = \frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{4}y_3$$

$$x_2 = \frac{7}{4}y_1 - \frac{5}{4}y_2 - \frac{3}{4}y_3$$

$$x_3 = -\frac{1}{2}y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3$$

Example 5.40. *Consider the transformation*

$$y_1 = -\frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3$$

$$y_2 = -\frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3$$

$$y_3 = -\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3$$

Verify whether it can transform $x_1^2 + x_2^2 + x_3^2$ *to* $y_1^2 + y_2^2 + y_3^2$, *write down the inverse transformation.*

Solution. Transformation is orthogonal.

EXERCISE 5.8

1. Are the following vectors linearly dependent? If so, find the relation between them.
   i. $x_1 = (1, 2, 4), x_2 = (2, -1, 3), x_3 = (0, 1, 2)$ and $x_4 = (-3, 7, 2)$
   ii. $x_1 = (1, 3, 4, 2), x_2 = (3, -5, 2, 2)$ and $x_3 = (2, -1, 3, 2)$
   iii. $x_1 = (2, -1, 3, 2), x_2 = (1, 3, 4, 2)$ and $x_3 = (3, -5, 2, 2)$
   iv. $x_1 = (2, 3, 1, -1), x_2 = (2, 3, 1, -2)$ and $x_3 = (4, 6, 2, 1)$
   v. $x_1 = (1, 2, 1), x_2 = (2, 1, 4)$ and $x_3 = (4, 5, 6)$
2. Prove that the set of vectors $(0, 2, -4), (1, -2, -1), (1, -4, 3)$ is linearly dependent.
3. Determine whether the following set of vectors are linearly dependent or linearly independent.
   i. $(1, -2, 1), (2, 1, -1), (7, -4, 1)$
   ii. $(1, 1, 1), (0, 4, 1), (3, 0, 1)$
   iii. $(2, 3, 1), (-1, 4, -2), (1, 18, -4)$
4. Prove that the set of four vectors $V_1 = (1, 0, -1), V_2 = (-1, 0, 0), V_3 = (1, 0, 1)$ and $V_4 = (2, 1, 3)$ is linearly dependent.
5. Find a if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.
6. Find P if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ P \\ -3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.
7. Show that the transformation $y_1 = x_1 - x_2 + x_3, y_2 = 3x_1 - x_2 + 2x_3, y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular. Find the inverse transformation.
8. Represent each of the transformations $x_1 = 3y_1 + 2y_2, x_2 = -y_1 + 4y_2, y_1 = z_1 + 2z_2$ and $y_2 = 3z_1$ by the use of the matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

Answers

1. i. Yes; $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$
   ii. Yes; $x_1 + x_2 - 2x_3 = 0$
   iii. Yes; $2x_1 - x_2 - x_3 = 0$
   iv. Yes; $5x_1 - 3x_2 - x_3 = 0$
   v. Yes; $2x_1 + x_2 - x_3 = 0$
3. i. Linearly dependent
   ii. Linearly independent
   iii. Linearly dependent
5. $a = 1$
6. $P = 2$

5.10 RANK OF A MATRIX

A non-negative integer 'r' is said to be the rank of a matrix 'A' if

1. There exist atleast one minor (square submatrix) of order 'r' which is not zero.
2. Every minor of matrix 'A' of order greater than 'r' is zero. Rank of A is denoted by $\rho(A)$.

Remarks: 1. $\rho(A) = 0$, when A is a zero matrix.

2. $\rho(A) \geq 1$, when $A \neq 0$.
3. If 'A' is a non-singular matrix of order 'n', then $\rho(A) = n$.
4. If 'A' is a singular matrix of order n, then $\rho(A) < n$.
5. If the order of a matrix 'A' is $m \times n$, then $\rho(A) \leq \min.\ (m, n)$.
6. Corresponding to every matrix 'A' of rank 'r', there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

For example: *Find the rank of the matrix* $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Given $$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $$|A| = 2(1) = 2 \neq 0$$

$\Rightarrow$ A is a non-singular matrix of order 3

$\Rightarrow$ $$\rho(A) = 3.$$

5.10.1 Another Way to Find the Rank of a Matrix

The rank of a matrix is equal to the number of non-zero rows in Echelon Form of that Matrix.

Remark: Non-zero row is that row in which atleast one element is not zero.

e.g. Consider a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Clearly, the given matrix 'A' is in Echelon form, which has two non-zero rows.
Hence the rank of $A = 2$ *i.e.* $\rho(A) = 2$.

SOME SOLVED EXAMPLES

Example 5.41. *Find the rank of the matrix* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

Solution. We have, $$A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1, R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$ $\quad A = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 \to 4R_3 - R_2$ $\quad A = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{bmatrix}$

The number of non-zero row is 3, therefore Rank $(A) = 3$

Example 5.42. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$.

Solution. Here, we have, $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 7 & 13 & -9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\quad A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The number of non-zero row is 2, therefore Rank $(A) = 2$.

Example 5.43. *For which value of 'b' the rank of the matrix* $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$ *is 2.*

Solution. Here we have, $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

$R_1 \to 2R_1$ $\quad A = \begin{bmatrix} 2 & 10 & 8 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\quad A = \begin{bmatrix} 2 & 10 & 8 \\ 0 & 3 & 2 \\ b-2 & 3 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2 \qquad A = \begin{bmatrix} 2 & 10 & 8 \\ 0 & 3 & 2 \\ b-2 & 0 & 0 \end{bmatrix}$

If rank of A is 2, then $b - 2$ must be zero

i.e., $\qquad b - 2 = 0 \quad \Rightarrow \quad b = 2.$

Example 5.44. *Find the rank of the matrix* $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

Solution. Here we have, $\quad A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$R_2 \to R_2 - \frac{3}{2}R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - \frac{9}{2}R_1$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & \frac{-1}{2} & -1 & \frac{-3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & \frac{-7}{2} & -7 & \frac{-21}{2} \end{bmatrix}$$

$R_3 \to R_3 - 2R_2,\ R_4 \to R_4 - 7R_2$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & \frac{-1}{2} & -1 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the number of non-zero row is 2 therefore Rank $(A) = 2$.

Example 5.45. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$.

Solution. We have, $\qquad A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2, R_4 \to R_4 - 3R_2$

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$
$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the number of non-zero row is 3, therefore Rank $(A) = 3$.

Example 5.46. *Find the rank of the matrix* $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Solution. Here, we have, $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2$
$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to R_3 - \frac{4}{5}R_2, R_4 \to R_4 - \frac{9}{5}R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 0 & 33/5 & 14/5 \\ 0 & 0 & 33/5 & 4/5 \end{bmatrix} \Rightarrow \quad R_4 \to R_4 - R_3, \quad A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 0 & 33/5 & 14/5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Rank = number of non-zero rows = 4, Rank $(A) = 4$.

Example 5.47. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

Solution. Here we have, $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_3$ $\qquad A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

$R_4 \to R_4 - R_2$ $\qquad A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad R_2 \leftrightarrow R_3, \quad A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank = number of non-zero rows = 3, Rank $(A) = 3$.

5.11 NORMAL FORM OF A MATRIX (CANONICAL FORM)

Every non-zero matrix of order $m \times n$ can be reduced by means of elementary row and column operation into equivalent matrix of any of the following forms:

i. $\begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$ $\qquad$ ii. $\begin{bmatrix} I_r \\ \cdots\cdots \\ 0 \end{bmatrix}$

iii. $[I_r \ : \ 0]$ $\qquad$ iv. $[I_r]$

Where I_r is the identity matrix of order r and 0 represent zero matrix of any order which is called its normal form or canonical form. The number r so obtained is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the first canonical form of A. Since both row and column transformation may be used here, the element 1 of the first row obtained can be moved in first column. Then both the first row and first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column and so on.

SOME SOLVED EXAMPLES

Example 5.48. *Find the rank of the matrix* $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ *by reducing it to normal form.*

Solution. Let $$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1, C_3 \to C_3 + C_1, C_4 \to C_4 - 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & -1 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

$C_3 \to C_3 + 2C_4, C_4 \to (-1)C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_4$ $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & : & 0 \\ 0 & : & 0 \end{bmatrix}$$

Which is required normal form.
Rank of Matrix $(A) = 2$.

Example 5.49. *Find the rank of the matrix* $A = \begin{bmatrix} 9 & 0 & 2 & 3 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$ *by reducing it to normal form.*

Solution. We have $$A = \begin{bmatrix} 9 & 0 & 2 & 3 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$$

$R_1 \to R_1\,(1/9)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 2/9 & 3/9 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_1(-4)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 2/9 & 3/9 \\ 0 & 1 & 5 & 6 \\ 0 & 5 & 19/9 & -12/9 \end{bmatrix}$

$C_3 \to C_3 + C_1\left(\dfrac{-2}{9}\right)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 3/9 \\ 0 & 1 & 5 & 6 \\ 0 & 5 & 19/9 & -12/9 \end{bmatrix}$

$C_4 \to C_4 + \left(\dfrac{-3}{9}\right)C_1$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 6 \\ 0 & 5 & 19/9 & -12/9 \end{bmatrix}$

$R_3 \to R_3 - R_2(5)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & \dfrac{-206}{9} & \dfrac{-282}{9} \end{bmatrix}$

$C_3 \to C_3 + C_2(-5)$
$C_4 \to C_4 + C_2(-6)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-206}{9} & \dfrac{-282}{9} \end{bmatrix}$

$R_3 \to R_3\left(\dfrac{-9}{206}\right)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \dfrac{141}{103} \end{bmatrix}$

$C_4 \to C_4 + C_3\left(\dfrac{-141}{103}\right)$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3 : 0]$

Which is required normal form.
Rank of Matrix $(A) = 3$.

Example 5.50. *Reduce the matrix* $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ *to normal form* $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Solution. We have $\qquad A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$C_2 \to C_2 + C_1, C_3 \to C_3 - 2C_1, C_4 \to C_4 + 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$R_4 \leftrightarrow R_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix}$$

$C_4 \to C_4 - 2C_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$
$R_4 \to R_4 - 5R_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix}$$

$C_3 \to \frac{-1}{2} C_3$
$C_4 \to \frac{-1}{8} C_4$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4 + 2R_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

Hence $\rho(A) = 4$.

Example 5.51. *Find the rank of the following matrix by reducing it to normal form* $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. We have
$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$
$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$
$C_3 \to C_3 + C_1$
$C_4 \to C_4 - 3C_1$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$C_3 \to C_3 + \frac{6}{7}C_2$
$C_4 \to C_4 - \frac{11}{7}C_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_4 \to R_4 + \frac{1}{2}R_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 + 2C_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{-1}{7}R_2$
$R_3 \to \frac{-1}{2}R_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of $A = 3$.

Example 5.52. *Reduce the matrix to normal form and find its rank if* $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

Solution. We have $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$$\begin{aligned} &R_2 \to R_2 - \frac{3}{2}R_1 \\ &R_3 \to R_3 - 2R_1 \\ &R_4 \to R_4 - \frac{9}{2}R_1 \end{aligned} \qquad A \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -7/2 & -7 & -21/2 \end{bmatrix}$$

$$\begin{aligned} &C_2 \to C_2 - \frac{3}{2}C_1 \\ &C_3 \to C_3 - 2C_1 \\ &C_4 \to C_4 - \frac{5}{2}C_1 \end{aligned} \qquad A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -7/2 & -7 & -21/2 \end{bmatrix}$$

$$\begin{aligned} &R_1 \to \frac{1}{2}R_1 \\ &R_2 \to -2R_2 \\ &R_3 \to -R_3 \\ &R_4 \to \frac{-2}{7}R_4 \end{aligned} \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} &R_3 \to R_3 - R_2 \\ &R_4 \to R_4 - R_2 \end{aligned} \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} &C_3 \to C_3 - 2C_2 \\ &C_4 \to C_4 - 3C_2 \end{aligned} \qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence its rank = 2.

Example 5.53. *Reduce the matrix A to its normal form, when* $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$.

Hence find the rank of A.

Solution. The given matrix is $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 + R_1$ $\quad A \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \Rightarrow$ $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - 4C_1$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$C_3 \leftrightarrow C_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix} \Rightarrow$ $R_3 \to R_3 - \frac{4}{5}R_2$, $R_4 \to R_4 - R_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0 & 16/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_2 \to 1/5R_2$
$R_3 \to 5/16R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$ $R_2 \to R_2 + \frac{5}{4}R_3$, $R_4 \to R_4 - R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \leftrightarrow C_4$ $\quad A \sim = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

Which is required normal form

Hence the rank of given matrix is = 3.

5.11.1 To Calculate P and Q where $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

If A is a matrix of order $m \times n$ then write $A = I_m A I_n$ where I_m and I_n are mth and nth order unit matrices respectively. The elementary row operation on A can be effected by premultiplication with corresponding elementary matrix *i.e.* application of the same to I_m or to the matrix obtained from I_m in subsequent steps. Similarly, application of an elementary column operation to A is equivalent to application of the same to I_n or the matrix obtained from I_n in subsequent steps. In the end when we get $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in place of A on left hand side, we have P in place of I_m and Q in place of I_n on the right hand side.

SOME SOLVED EXAMPLES

Example 5.54. *Find the non-singular matrices P and Q such that PAQ is in normal form where* $A = \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}$.

Solution. We have $\quad [A]_{2\times 3} = I_2 \,.\, A \,.\, I_3$

$$\begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to \frac{1}{2}R_1 \qquad \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 + R_1 \qquad \begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 + 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \to \frac{1}{3}C_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & 3 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 \to C_3 + C_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & 8/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[I_2, 0] = PAQ$$

Hence $P = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1/3 & 8/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 5.55. *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$. *Find two non-singular matrices P and Q such that PAQ = I.*

Solution. $[A]_{3\times 3} = I_3\, A\, I_3$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to R_1 - R_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to -C_2$ $$\begin{bmatrix}1&0&0\\0&3&4\\0&1&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

$R_2 \leftrightarrow R_3$ $$\begin{bmatrix}1&0&0\\0&1&1\\0&3&4\end{bmatrix}=\begin{bmatrix}1&-1&0\\0&0&1\\-2&3&0\end{bmatrix}A\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

$R_3 \to R_3 - 3R_2$ $$\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\0&0&1\\-2&3&-3\end{bmatrix}A\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

$C_3 \to C_3 - C_2$ $$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\0&0&1\\-2&3&-3\end{bmatrix}A\begin{bmatrix}1&0&0\\0&-1&1\\0&0&1\end{bmatrix}$$

$$I_3 = PAQ$$

i.e. $$P=\begin{bmatrix}1&-1&0\\0&0&1\\-2&3&-3\end{bmatrix}\qquad Q=\begin{bmatrix}1&0&0\\0&-1&1\\0&0&1\end{bmatrix}$$

Example 5.56. *Find the non singular matrices P and Q such that PAQ is in the normal form when* $A=\begin{bmatrix}3&1&2&1\\1&4&6&1\\2&-3&1&-2\end{bmatrix}$.

Solution. $$A = I_3\,A\,I_4$$

$$\begin{bmatrix}3&1&2&1\\1&4&6&1\\2&-3&1&-2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $$\begin{bmatrix}1&4&6&1\\3&1&2&1\\2&-3&1&-2\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$ $$\begin{bmatrix}1&4&6&1\\0&-11&-16&-2\\0&-11&-11&-4\end{bmatrix}=\begin{bmatrix}0&1&0\\1&-3&0\\0&-2&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2 \to C_2 - 4C_1$, $C_3 \to C_3 - 6C_1$, $C_4 \to C_4 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -11 & -16 & -2 \\ 0 & -11 & -11 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -11 & -11 & -4 \\ 0 & -11 & -16 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & -3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2\,(-1/11)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4/11 \\ 0 & -11 & -16 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2/11 & -1/11 \\ 1 & -3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 11R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4/11 \\ 0 & 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ 1 & -1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -6 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2,\ C_4 \to C_4 - \dfrac{4}{11} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ 1 & -1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -2 & 5/11 \\ 0 & 1 & -1 & -4/11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to \dfrac{-1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2/5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -2 & 5/11 \\ 0 & 1 & -1 & -4/11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 + C_3 \left(\dfrac{2}{5}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & +1 & -1 & -42/55 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3, 0] = PAQ$$

Hence

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & 1 & -1 & -42/55 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Verification:

$$PAQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/11 & -1/11 \\ -1/5 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 2 & -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & 1 & -1 & -42/55 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 6 & 1 \\ 0 & 1 & 1 & 4/11 \\ 0 & 0 & 1 & -2/5 \end{bmatrix} \begin{bmatrix} 1 & -4 & -2 & -19/55 \\ 0 & 1 & -1 & -42/11 \\ 0 & 0 & 1 & 2/5 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

EXERCISE 5.9

Find the rank of following matrices:

1. $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & -1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 5 & 4 & 1 & 6 \\ 2 & 3 & -1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

Find the rank of following matrices after reducing them to normal form:

8. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

11. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Find two non singular matrices P and Q such PAQ is in normal form for the matrix A, where A is

12. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Answers

1. 2 **2.** 2 **3.** 2 **4.** 2

5. 2 **6.** 2 **7.** 2 **8.** 2

9. 3 **10.** 3 **11.** 3

12. $P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

13. $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & -1/3 & 1/6 \end{bmatrix} Q = \begin{bmatrix} 1 & 4/7 & 9/119 & 9/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$

14. $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

INTERESTING FACTS

- Rank of a matrix even has its application in the domain of data mining and bioinformatics.
- In social sciences, the user preferences for countries are represented in the form of matrices.
- In biological sciences, the level of gene expression is represented using matrices.
- In medical sector, cancer and its subtypes are discovered from their molecular data using matrices.

VIDEO REFERENCES

Rank of a Matrix

System of Linear Equations-II

System of Linear Equations

APPLICATIONS TO REAL LIFE

- These are vitally used to build mathematical and computer programming modelling.
- If some unknown data needs to be recovered in matrix form, let's say in cyber security space, and it is known that it has a low rank, then it can be recovered very efficiently and easily.
- The rank even gives an idea about the dimension of the image. For example, the mapping of 3D space into a 2D plane will not have a "***full rank***".

5.12 RANK-NULLITY THEOREM (In terms of Matrices)

Nullity = No. of columns – Rank of a Matrix

Statement: Let A be an $m \times n$ matrix with rank 'r' and nullity l, then according to rank-nullity theorem, rank (A) + nullity $(A) = n$

i.e., $$r + l = n,$$

where r = rank of matrix

l = nullity of matrix

n = no. of columns

For example: Find the nullity of the given matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{bmatrix}_{2\times 4}$

Solution. First we find rank, using any method discussed earlier,

so apply $R_2 \to R_2 - 2R_1$

We have $$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

We can see, $\rho(A) = 2$ (no. of non-zero rows)

then as per theorem,

$$\text{rank} + \text{nullity} = \text{no. of columns}$$

so $$2 + \text{nullity} = 4$$

$\Rightarrow$ $$\text{nullity} = 4 - 2 = 2$$

SOME SOLVED EXAMPLES

Example 5.57. *For the given matrix*

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{bmatrix}$$

Find the nullity.

Solution. Given, $$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{bmatrix}$$

Apply $$R_3 \to R_3 + R_1,$$
$$R_2 \to R_2 - 3R_1$$

We have,
$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \end{bmatrix}$$

then, apply $R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have, Rank of $A = \rho(A) = 2$

So nullity = no. of columns – Rank of matrix

$= 4 - 2 = 2$

Example 5.58. *For the given matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5}$$

Find nullity.

Solution. As the given matrix A is in Echelon form,

therefore $\rho(A) = 3$

We know, rank (A) + nullity (A)= no. of columns

So, $3 +$ nullity $(A) = 5$

nullity $(A) = 2$

Note: In general, rank $(A) =$ rank (A^T)

But nullity $(A) \neq$ nullity (A^T)

We can see, in the given example:

Suppose
$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5}$$

then rank $(A) = 3$,

$A^T = 5 \times 4$ (matrix)

then rank $(A^T) = 3$ (As rank of matrix A and its transpose is same)

But nullity $(A) = 5 - 3 = 2$ (As no. of columns are 5)

and nullity $(A^T) = 4 - 3 = 1$ (As no. of columns are 4)

Example 5.59. *The nullity of the matrix*

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}_{3\times 4}$$

Solution. Given $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

First, find Rank (A)

Apply $R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 7R_1$

then, $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$

$R_3 \to R_3 + \dfrac{R_2}{2}$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1/3, R_2 \to R_2/42$

$$= \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & 2/3 & 29/21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\rho(A) = 2$

So nullity $(A) = 4 - 2 = 2$ (As the no. of columns are 4)

5.13 LINEAR SYSTEM OF EQUATIONS

Suppose in my neighbourhood, there is an eccentric shopkeeper. He is convinced that some Indians eat more wheat than rice and some Indians eat more rice than wheat. So he offers only two standard packets. The first packet, call it N, has 5 kg of wheat and 2 kg of rice, whereas the second packet, call it S, has 2 kg of wheat and 5 kg of rice. Let us invent a shorthand. Whenever we write (m, n), we mean m kg of wheat and n kg of rice. Now if I buy 3 packets of N, it means that I am buying 15 kg of wheat and 6 kg of rice, *i.e.*, $3N = 3(5, 2) = (15, 6)$.

Similarly, 2 packets of S means 4 kg of wheat and 10 kg of rice, *i.e.*, $2S = 2(2, 5) = (4, 10)$.

If I buy one of each of the packets , then I would have bought 7 kg of wheat and 7 kg of rice, that is,

$$N + S = (5, 2) + (2, 5) = (5 + 2, 2 + 5) = (7, 7).$$

Thus I need m packets of N or n packets of S or both, there is no problem. Suppose I need 19 kg of wheat and 16 kg of rice. What shall I do? I need to buy x packets of N and y packets of S so that $x(5, 2) + y(2, 5) = (19, 16)$.

That is, $(5x, 2x) + (2y, 5y) = (19, 16)$ or $(5x + 2y, 2x + 5y) = (19, 16)$. Thus I end up solving a system of linear equations

$$5x + 2y = 19$$
$$2x + 5y = 16.$$

Explanation in terms of *n* equation in *n*-unknowns:

A linear system of n equations in n unknowns $x_1, x_2 \ldots\ldots x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \ldots\ldots\ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots\ldots\ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + \ldots\ldots\ldots\ldots + a_{nn}x_n &= b_n \end{aligned} \qquad \ldots(1)$$

where aij $(1 \le i, j \le n)$ are the known coefficients, b_i $(1 \le i \le n)$ are given numbers.

Note: The system is called homogeneous if all the *bi's* are zero.

Otherwise it is non-homogeneous.

In the matrix notation, the system (1) can be written as

$$AX = B$$

Where
$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots\ldots\ldots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

The augmented matrix of the system (1) can be written as

$$C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots\ldots\ldots & a_{nn} & b_n \end{bmatrix}$$

A solution of (1) is the set of number $x_1 \ldots\ldots\ldots\ldots x_n$ that satisfy all the n equations.

Remark: Augmented Matrix: An augmented matrix from a system of equations is a matrix of numbers in which each row represents the constants from one equation (both the coefficient and the constant on the other side of the equal sign) and each column represents all the coefficients for a single variable.

5.13.1 Types of Linear Equations

A. Non-Homogeneous Equations

B. Homogeneous Equations

5.13.1.1 Non-Homogeneous Systems

i. Consistent: A system of equations is said to be consistent, if they have one or more solution *i.e.*

$x + 2y = 4$	$x + 2y = 4$
$3x + 2y = 2$	$3x + 6y = 12$
Unique solution	Infinite solution

ii. Inconsistent: If a system of equation has no solution, it is said to be inconsistent.

$$x + 2y = 4$$
$$3x + 6y = 5$$

Consistency of a System of Linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n &= b_m \end{aligned}$$

In matrix notation, these of equations are written as:

$$AX = B$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots\ldots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented matrix is $$C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & : & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & : & b_m \end{bmatrix}$$

a. **Consistent equation:**

If Rank A = Rank [A : B]

i. Unique solution: Rank A = Rank $[A : B] = n$ (no. of unknowns)

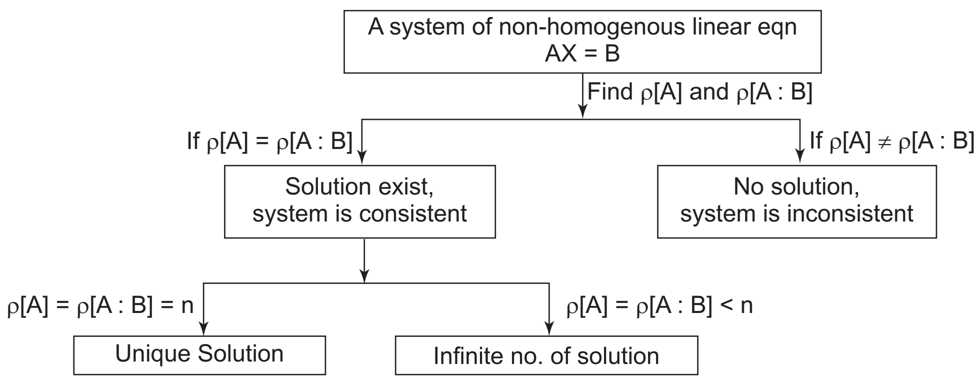
ii. Infinite solution: Rank A = Rank $[A : B] < n$ (no. of unknowns)

b. **Inconsistent equation:**

If Rank $A \neq$ Rank $[A : B]$

Thus we can say that the system of linear equations are either non-homogeneous or Homogeneous. it depends on b_i, then system of non-homogeneous equations are either consistent or inconsistent according to which system has unique solution or infinitely many solution.

In brief



- n = no. of unknowns.

SOME SOLVED EXAMPLES

Example 5.60. *Show that the equations*

$$2x + 6y = -11$$
$$6x + 20y - 6z = -3$$
$$6y - 18z = -1$$

are not consistent.

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

$$AX = B$$

where $$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

Augmented matrix $[A : B]$ is

$$= \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ $$\begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$ $$\begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix}$$

$$\rho(A) = 2 \text{ and } \rho(A : B) = 3$$

$\therefore$ $$\rho(\text{A}) \neq \rho(A : B)$$

Hence, the system of equations are not consistent.

Example 5.61. *Test for consistency and solve*

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$AX = B$$

where $$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Augmented matrix $[A : B]$ is

$$\begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$R_1 \to R_1\left(\frac{1}{5}\right)$ $$\begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 7R_1$ $$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & : & \frac{33}{5} \\ 0 & \frac{-11}{5} & \frac{1}{5} & : & \frac{-3}{5} \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{11}R_2$ $$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & : & \frac{33}{5} \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = 2 \text{ and } \rho[A : B] = 2$$

$\therefore$ $$\rho(A) = \rho[A : B]$$

Hence, the given system of equations are consistent.

But $\rho(A) = \rho[A : B] < n$ (no. of unknowns)

So, its solutions are infinite.

$$x + \frac{3}{5}y + \frac{7}{5}z = \frac{4}{5} \quad \text{...(1)}$$

$$\frac{121}{5}y - \frac{11}{5}z = \frac{33}{5} \quad \Rightarrow \quad 11y - z = 3$$

Let $z = k$, then $11y = 3 + k$

$$y = \frac{3}{11} + \frac{k}{11}$$

Put value of y and z in (1)

$$x + \frac{3}{5}\left(\frac{3}{11} + \frac{k}{11}\right) + \frac{7}{5}k = \frac{4}{5}$$

$$x + \frac{9}{55} + \frac{3k}{55} + \frac{7k}{5} = \frac{4}{5}$$

$$x = \frac{4}{5} - \frac{9}{55} - \left(\frac{3k}{55} + \frac{7k}{5}\right)$$

$$x = \frac{7}{11} - \frac{16}{11}k$$

Example 5.62. *Test for consistency of the following system of equations:*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ 6x_1 + 7x_2 + 8x_3 + 9x_4 &= 10 \\ 11x_1 + 12x_2 + 13x_3 + 14x_4 &= 15 \\ 16x_1 + 17x_2 + 18x_3 + 19x_4 &= 20 \\ 21x_1 + 22x_2 + 23x_3 + 24x_4 &= 25 \end{aligned}$$

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \\ 21 & 22 & 23 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 25 \end{bmatrix}$$

$$AX = B$$

The augmented matrix $[A : B]$ is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 6 & 7 & 8 & 9 & : & 10 \\ 11 & 12 & 13 & 14 & : & 15 \\ 16 & 17 & 18 & 19 & : & 20 \\ 21 & 22 & 23 & 24 & : & 25 \end{bmatrix}$$

$R_2 \to R_2 - 6R_1$
$R_3 \to R_3 - 11R_1$
$R_4 \to R_4 - 16R_1$
$R_5 \to R_5 - 21R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & -5 & -10 & -15 & : & -20 \\ 0 & -10 & -20 & -30 & : & -40 \\ 0 & -15 & -30 & -45 & : & -60 \\ 0 & -20 & -40 & -60 & : & -80 \end{bmatrix} \Rightarrow (-1)^4 \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 5 & 10 & 15 & : & 20 \\ 0 & 10 & 20 & 30 & : & 40 \\ 0 & 15 & 30 & 45 & : & 60 \\ 0 & 20 & 40 & 60 & : & 80 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 3R_2$
$R_5 \to R_5 - 4R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 5 & 10 & 15 & : & 20 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = 2 \text{ and } \rho[A : B] = 2$$

$\therefore$ $$\rho(A) = \rho[A : B]$$

Hence, the given system of equations are consistent.

But $\rho(A) = \rho[A : B] < n$

So, its solutions are infinite.

Example 5.63. *For what value of k, the system*

$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2 \text{ has a solution.}$$

Solution. The given system of equations in matrix form are

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

$$AX = B$$

Augmented matrix $[A : B]$

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -3 & 6 & : & k^2 - 4 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & 0 & 0 & : & k^2 - 3k + 2 \end{bmatrix}$$

If the given system has solution, then

$$\rho(A) = \rho[A : B]$$

and $\rho[A : B] = 2$ if $k^2 - 3k + 2 = 0$

$$k^2 - 2k - k + 2 = 0$$
$$(k - 2)\,(k - 1) = 0$$
$$k = 2, k = 1.$$

Case I: When $k = 1$, we have

$$x + y + z = 1 \quad \text{...(1)}$$

$$-y + 2z = 1 - 2 = -1 \quad \text{...(2)}$$

Let $z = \lambda$

Putting value of $z = \lambda$ in (2) $y = 2\lambda + 1$

Putting the value of y and z in (1)

$$x + (2\lambda + 1) + \lambda = 1$$

$$x + 3\lambda + 1 = 1$$

$$x = -3\lambda$$

Case II: When $k = 2$

$$x + y + z = 1 \quad \text{...(3)}$$

$$-y + 2z = 2 - 2 = 0 \quad \text{...(4)}$$

Let $z = c$

Putting the value of z in (4)

$$-y + 2c = 0$$

$$y = 2c$$

Putting the value of y and z in (3)

$$x + 2c + c = 1$$

$$x = 1 - 3c$$

Example 5.64. *Investigate the values of λ and μ so that the equations:*

$$2x + 3y + 5z = 9$$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

have i. *no solution*

ii. *a unique solution*

iii. *an infinite no. of solution.*

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Augmented matrix is $[A : B]$
$$\begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - \frac{7}{2}R_1$

$R_3 \to R_3 - R_1$
$$\begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & : & \frac{-47}{2} \\ 0 & 0 & \lambda - 5 & : & \mu - 9 \end{bmatrix}$$

i. No solution: Rank $A \neq$ Rank $(A : B)$

$$\lambda - 5 = 0 \qquad \mu - 9 \neq 0$$
$$\lambda = 5 \qquad \mu \neq 9$$

ii. An infinite no. of solutions: Rank A = Rank $(A : B) < n$

$$\lambda - 5 = 0 \qquad \mu - 9 = 0$$
$$\lambda = 5 \qquad \mu = 9$$

iii. A unique solution Rank A = Rank $(A : B) = n$

$$\lambda - 5 \neq 0$$
$$\lambda \neq 5, \mu \text{ is arbitrary}$$

Example 5.65. *Determine for what value of λ and μ the following equation have*

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

i. *no solution*
ii. *a unique solution*
iii. *infinite number of solution.*

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX = B$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

i. No solution: $\rho(A) \neq \rho[A : B]$

$$\lambda - 3 = 0 \qquad \mu - 10 \neq 0$$
$$\lambda = 3 \qquad \mu \neq 10$$

ii. A unique solution: $\rho(A) = \rho[A : B] = n$

$$\lambda - 3 \neq 0$$
$$\lambda \neq 3, \mu \text{ is arbitrary.}$$

iii. infinite solution: $\rho(A) = \rho[A : B] < n$

$$\lambda - 3 = 0 \qquad \mu - 10 = 0$$

$$\lambda = 3 \qquad \mu = 10$$

Example 5.66. *Show that the equation*

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

have no solution unless $a + b + c = 0$. In which case they have infinitely many solution? Find these solutions when $a = 1$, $b = 1$ and $c = -2$.

Solution. The given system of equations in matrix form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$AX = B$$

The augmented matrix is $[A : B]$

$$\begin{bmatrix} -2 & 1 & 1 & : & a \\ 1 & -2 & 1 & : & b \\ 1 & 1 & -2 & : & c \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$$\begin{bmatrix} 1 & -2 & 1 & : & b \\ -2 & 1 & 1 & : & a \\ 1 & 1 & -2 & : & c \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - R_1$
$$\begin{bmatrix} 1 & -2 & 1 & : & b \\ 0 & -3 & 3 & : & a + 2b \\ 0 & 3 & -3 & : & c - b \end{bmatrix}$$

$R_3 \to R_3 + R_2$
$$\begin{bmatrix} 1 & -2 & 1 & : & b \\ 0 & -3 & 3 & : & a + 2b \\ 0 & 0 & 0 & : & a + b + c \end{bmatrix}$$

Case I: If $a + b + c \neq 0$

$$\rho(A) = 2 \text{ and } \rho(A : B) = 3$$

$\therefore$ $\quad \rho(A) \neq \rho(A : B)$

Hence, the system being inconsistent, have no solution.

Case II: If $a + b + c = 0$

$$\rho(A) = 2, \rho(A : B) = 2$$

$\therefore$ $\quad \rho(A) = \rho(A : B)$

Hence, the given system of equations are consistent.

But $\quad \rho(A) = \rho(A : B) < n$

So, its has infinite no. of solution.

Case III: On putting $a = 1, b = 1, c = -2$

$$\begin{bmatrix} 1 & -2 & 1 & : & 1 \\ 0 & -3 & 3 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x - 2y + z = 1 \quad \text{...(1)}$$

$$-3y + 3z = 3 \quad \text{...(2)}$$

$$-y + z = 1$$

Put $z = k$, $\quad y = k - 1$

Put the value of y and z in (1)

$$x - 2(k - 1) + k = 1$$

$$x - 2k + 2 + k = 1$$

$$x - k + 2 = 1$$

$$x = k - 1$$

$$x = k - 1, \quad y = k - 1, \quad z = k$$

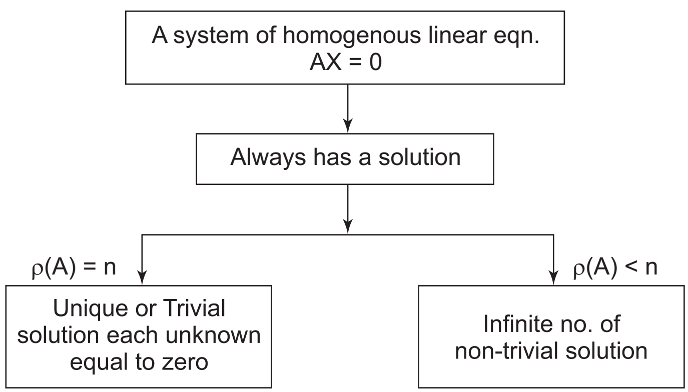
5.13.1.2 Homogeneous Equations

For a system of homogeneous linear equations $AX = 0$.

i. $X = 0$ is always a solution. This solution in which each unknown has the value zero is called the null solution or the Trivial solution. Thus a Homogeneous system is always consistent.
   - A system of Homogeneous linear equations has either the trivial solution or infinite no. of solutions.

ii. If $\rho(A)$ = no. of unknowns the system has only trivial solution.

iii. If $\rho(A)$ < no. of unknowns, the system has infinite no. of non-trivial solutions.



Example 5.67. *Determine the value of b such that the system of Homogeneous equations*

$$2x + y + 2z = 0$$

$$x + y - 3z = 0$$

$$4x + 3y + bz = 0$$

has i. *Trivial solution*

ii. *Non-trivial solution. Find Non-trivial solution using matrix method.*

Solution. i. ***For Trivial solution:*** We know that $x = 0, y = 0\ z = 0$, so b can have any value.

ii. ***For Non-Trivial solution:*** The given system of equation in matrix form is

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & -3 \\ 4 & 3 & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

Augmented matrix is $[A : B]$

$$\begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 1 & 1 & -3 & : & 0 \\ 4 & 3 & b & : & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$$\begin{bmatrix} 1 & 1 & -3 & : & 0 \\ 2 & 1 & 2 & : & 0 \\ 4 & 3 & b & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$$\begin{bmatrix} 1 & 1 & -3 & : & 0 \\ 0 & -1 & 8 & : & 0 \\ 0 & -1 & b+12 & : & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\begin{bmatrix} 1 & 1 & -3 & : & 0 \\ 0 & -1 & 8 & : & 0 \\ 0 & 0 & b+4 & : & 0 \end{bmatrix}$$

For Non-trivial solution $\rho[A] = \rho[A : B] < n$

$$b + 4 = 0$$
$$b = -4$$

Example 5.68. *Find the value of k such that system of equations*

$$x + ky + 3z = 0$$
$$4x + 3y + kz = 0$$
$$2x + y + 2z = 0$$

has non trivial solution.

Solution. The given system of equation in matrix form is

$$\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

The augmented matrix is $[A : B]$

$$\begin{bmatrix} 1 & k & 3 & : & 0 \\ 4 & 3 & k & : & 0 \\ 2 & 1 & 2 & : & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \qquad \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 4 & 3 & k & : & 0 \\ 1 & k & 3 & : & 0 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \end{array} \qquad \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & k-\frac{1}{2} & 2 & : & 0 \end{bmatrix}$$

$$R_3 \to R_3 - \left(k-\frac{1}{2}\right)R_2 \qquad \begin{bmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & 0 & 2-\left(k-\frac{1}{2}\right)(k-4) & : & 0 \end{bmatrix}$$

For Non-trivial solution $\rho[A] = \rho[A : B] < n$

$$2 - \left(k - \frac{1}{2}\right)(k-4) = 0$$

$$2 - k^2 + 4k + \frac{k}{2} - 2 = 0$$

$$-k^2 + \frac{9}{2}k = 0$$

$$k\left(-k + \frac{9}{2}\right) = 0$$

$$k = 0, k = \frac{9}{2}.$$

EXERCISE 5.10

Check the consistency of the following system of equations. Also find the solution set:

1. $x + 2y - z = 3$
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$
$x - y + z = -1$

2. $x + 3y - z = 4$
$2x + y + z = 7$
$2x - 4y + 4z = 6$
$3x + 4y = 1$

3. $x_1 + 2x_2 - x_3 = 6$
$3x_1 - x_2 + 2x_3 = 3$
$4x_1 - 3x_2 + x_3 = 9$

4. $x - 4y - 3z = -16$
$2x + 7y + 12z = 48$
$4x - y + 6z = 16$
$5x - 5y + 3z = 0$

5. Discuss the consistency of the equation
$x + 2y + 3z + 4t = 0$
$2x + 3y + 4z - 1 = 0$
$3x + 4y + t = 2$
$4x + z + 2t = 3$ Find the solution set if consistent.

6. For what value of λ will the equations

$$3x - y + \lambda z = 1$$
$$2x + y + z = 2$$
$$x + 2y - \lambda z = -1 \text{ Fail to have a unique solution.}$$

Will the equations have any solution for this value of λ?

7. Use the test of rank to show that the following system of equations is inconsistent:

$$2x - y + z = 4$$
$$3x - y + z = 6$$
$$4x - y + 2z = 7$$
$$-x + y - z = 9$$

8. Show that the following equations are consistent and solve them:

$$x + 2y - 5z = -9$$
$$3x - y + 2z = 5$$
$$2x + 3y - z = 3$$
$$4x - 5y + z = -3$$

9. Solve the system of equations:

$$\lambda x + 2y - 2z = 1$$
$$4x + 2\lambda y - z = 2$$
$$6x + 6y + \lambda z = 3 \text{ considering specially the case when } \lambda = 2$$

10. For what value of a and b the equation

$$x + y + 5z = 0$$
$$x + 2y + 3az = b$$
$$x + 3y + az = 1 \text{ have}$$

i. No solution
ii. unique solution
iii. infinitely many solutions

Solve the following system of equations:

11.
$$x - y + z = 0$$
$$-3x + y - 4z = 0$$
$$7x - 3y - 9z = 0$$
$$4x - 2y + 5z = 0$$

12.
$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

13.
$$2w + 3x - y - z = 0$$
$$4w - 6x - 2y + 2z = 0$$
$$-6w + 12x + 3y - 4z = 0$$
$$8w - 24x - 4y + 8z = 0$$

14.
$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y + 13z + 3w = 0$$

15. Find the value of k such that following system of equations has a non-trivial solution:

$$(3k - 8)x + 3y + 3z = 0$$
$$3x + (3k - 8)y - 3z = 0$$
$$3x + 3y + (3k - 8)z = 0$$

16. Show that the only real value of λ for which the equations:

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$ have a non-zero solution 6.

Answers

1. $-1, 4, 4$ **2.** Not consistent **3.** $-1, 4, 4$

4. $\frac{17}{5} - \frac{4}{5}k, \frac{1}{5} + \frac{3}{5}k, k$ **5.** Consistent, $\left(\frac{9}{11}, \frac{-1}{11}, \frac{-1}{11}, \frac{-1}{11}\right)$ **6.** Inconsistent. $\lambda = -\frac{7}{2}$

8. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ **9.** $x = \frac{1}{2} - k, y = k, z = 0$

10. i. $a = 1, b \neq 1/2$ ii. $a \neq 1, b \in R$ iii. $a = 1, b = 1/2$

11. $x = y = z = 0$ **12.** $x = \frac{-10}{7}k, y = \frac{8}{7}k, z = k$

13. $x = \frac{1}{3}k_1,\ y = k_2, z = k_1,\ w = \frac{1}{2}k_2$ **14.** $x = 11k_1 + 6k_2, y = -8k_1 - 3k_2, z = k_1, w = k_2$

15. Non trivial solution if $k = 11/3$ or $k = 2/3$

5.14 EIGEN VALUES AND EIGEN VECTORS OF A MATRIX

Let A be a square matrix and 'X' is a column vector, then the matrix equation

$$AX = \lambda X$$

is equivalent to the n equations

$$\left.\begin{aligned} &(a_{11} - \lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ &a_{21}x_1 + (a_{22} - \lambda)x_2 + \ldots + a_{2n}x_n = 0, \\ &\quad\vdots \\ &\quad\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn} - \lambda)x_n = 0 \end{aligned}\right\} \qquad \ldots(1)$$

The above system of linear homogeneous equations (1) in 'n' unknowns always has the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

In order to find a non-trivial solution, it is necessary that the determinant of the coefficients in (1) vanishes; *i.e.*

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \ldots(2)$$

It is clear that eqn. (2) is of degree n in λ's This is called the characteristic equation of the matrix A. We can write it as

$$\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_{n-1}\lambda + p_n = 0 \qquad \ldots(3)$$

In particular, $p_1 = -(a_{11} + a_{22} + \ldots + a_{nn}) = -\text{(trace } A)$ ...(4)

and $p_n = (-1)^n \,|\, a_{ij} \,| = (-1)^n$ (det. A) ...(5)

The left hand side of eqn. (3) is called the characteristic polynomial.

Equation (3) will have n-roots. These are called characteristic (or latent) roots or eigen values of the matrix A. Corresponding to each root, eqn. (1) will have a non-zero solution.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

which is known as characteristic vector or eigen vector.

If the eigen values of A are $\lambda_1, \lambda_2, \ldots \lambda_n$, we can write the characteristic equation as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots (\lambda - \lambda_n) = 0,$$

or $$\lambda^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)\lambda^{n-1} + \ldots + (-1)^n \lambda_1\lambda_2 \ldots \lambda_n = 0$$

Comparing with eqn. (3), we can see that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = -p_1 = \text{trace } A, \qquad \text{by (4)}$$

and $$\lambda_1\lambda_2 \ldots \lambda_n = (-1)^n p_n = \text{det. } (A), \qquad \text{by (5)}$$

Remarks:

1. Eigen values are also called as characteristic value, proper value, latent value or spectral value.
2. Similarly, Eigen vectors are also called as characteristic vector, proper vector, latent vector or spectral vector.

5.14.1 Properties of Eigen Values

a. The eigen value of a square matrix A and its transpose A' are same.

b. The sum of the eigen values of a matrix is same as the sum of the elements on the principle diagonal.

c. The product of the eigen values of a matrix A is equal to det. (A) *i.e.*, $|A|$

d. If λ is the eigen value of a non-singular matrix A, then $1/\lambda$ is an eigen value of A^{-1}.

e. If λ is an eigen value of an orthogonal matrix, then $\dfrac{1}{\lambda}$ is also its eigen value.

f. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a +ve integer)

g. The eigen values of idempotent matrix are either zero or unity.

h. Eigen values of the triangular matrix and diagonal matrix are same as the diagonal elements of that matrix.

Proof. d. If X be the given eigen vector corresponding to eigen value λ of A, then

$$AX = \lambda X$$

Pre-multiplying both sides by A^{-1}

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$\Rightarrow \quad (A^{-1}A)X = \lambda A^{-1}X \qquad \Rightarrow \quad IX = \lambda A^{-1}X$$

$$\Rightarrow \qquad \frac{1}{\lambda}X = A^{-1}X \qquad [\because \quad IX = X]$$

$\Rightarrow$ $\dfrac{1}{\lambda}$ is the eigen value of A^{-1}.

Proof. f. If X be the eigen vector of the eigen value λ of matrix A, then

$$AX = \lambda X$$

Pre multiplying both sides by 'A'

$$A(AX) = A(\lambda X) \quad \Rightarrow \quad (AA)X = \lambda(AX)$$

$$\Rightarrow \qquad A^2X = \lambda AX$$

$$\Rightarrow \qquad A^2X = \lambda^2 X \qquad (\text{As } AX = \lambda X)$$

Again pre multiplying both sides by 'A', we have, in a similar way

$$A^3X = \lambda^3 X$$

Continue in the same manner, we have

$$A^m \,.\, X = \lambda^m X$$

which shows that λ^m is the eigen value of A^m $(m > 0)$

Thus, we can say that if $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigen values of A, then $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$ will be the eigen values of A^m.

SOME SOLVED EXAMPLES

Example 5.69. *Find the characteristic polynomial of matrix* $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix}$.

Solution. Characteristic matrix of A is

$$[A - \lambda I] = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & 3 \\ -2 & 1 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & 3 \\ -2 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 15$$

Example 5.70. *Find all the eigen values and eigen vectors of matrix* $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Solution. The characteristic equation of the given matrix is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (8-\lambda)\,((7-\lambda)\,(3-\lambda)-16)+6(-6(3-\lambda)+8)+2(24-2(7-\lambda))=0$

$\Rightarrow \quad (8-\lambda)\,(\lambda^2-10\lambda+5)+6(-10+6\lambda)+2(10+2\lambda)=0$

$\Rightarrow \quad 8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda-60+36\lambda+20+4\lambda=0$

$\Rightarrow \quad -\lambda^3+18\lambda^2-45\lambda=0$

$\Rightarrow \quad \lambda^3-18\lambda^2+45\lambda=0$ [Characteristic equation]

$\Rightarrow \quad \lambda(\lambda^2-18\lambda+45]=0$

$\Rightarrow \quad \lambda=0, \lambda^2-18\lambda+45=0$

$\Rightarrow \quad \lambda^2-15\lambda-3\lambda+45=0$

$\Rightarrow \quad \lambda(\lambda-15)-3(\lambda-15)=0$

$\Rightarrow \quad \lambda-3=0; \lambda-15=0$

$\Rightarrow \quad \lambda=3; \lambda=15; \lambda=0$

Eigen values are $\lambda = 0, 3, 15$

To find eigen vector : $[A-\lambda I]\,X=0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

for $\lambda = 0$, $$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad 8x_1-6x_2+2x_3=0$ ...(1)

$\Rightarrow \quad -6x_1+7x_2-4x_3=0$ ...(2)

$\Rightarrow \quad 2x_1-4x_2+3x_3=0$ ...(3)

Multiply eqn. (1) by 2 and adding to (2), we have

$10x_1-5x_2=0$

$\Rightarrow \quad 10x_1=5x_2$

$\Rightarrow \quad x_2=2x_1$ ...(4)

Multiply eqn. (3) by 3 and adding to (2), we have

$-5x_2+5x_3=0$

$\Rightarrow \quad -5x_2=-5x_3$

$\Rightarrow \quad x_2=x_3$ ...(5)

From (4) and (5), we get

$$2x_1=x_2=x_3 \quad \Rightarrow \quad \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

Eigen vector corresponding to $\lambda = 0$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

for $\lambda = 3$, $$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $5x_1 - 6x_2 + 2x_3 = 0$ ...(6)

$\Rightarrow$ $-6x_1 + 4x_2 - 4x_3 = 0$ ...(7)

$\Rightarrow$ $2x_1 - 4x_2 + 0.x_3 = 0$ ...(8)

From (8), $2x_1 = 4x_2$

$\Rightarrow$ $x_1 = 2x_2$

Put $x_1 = 2x_2$ in eqn. (7), we get

$\Rightarrow$ $x_3 = -2x_2$

Thus, $x_1 = 2x_2 = -x_3$

or $\dfrac{x_1}{2} = \dfrac{x_2}{1} = \dfrac{x_3}{-2}$

Eigen vectors are corresponding to $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

for $\lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $-7x_1 - 6x_2 + 2x_3 = 0$ ...(9)

$-6x_1 - 8x_2 - 4x_3 = 0$ ...(10)

$2x_1 - 4x_2 - 12x_3 = 0$ ...(11)

Multiply eqn. (11) by 3 and adding in eqn. (10), we have

$$x_2 = -2x_3$$

Multiply eqn. (9) by 2 and adding in eqn. (10), we have

$$x_1 = -x_2$$

$\therefore \qquad x_1 = -x_2 = 2x_3$

or $\qquad \dfrac{x_1}{2} = \dfrac{x_2}{-2} = \dfrac{x_3}{1}$

Thus, the eigen vectors corresponding to $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Example 5.71. *Find all the eigen values, eigen vectors and of eigen basis of* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution. Let $\qquad A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristic equation is, $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (-2-\lambda)\,(-(1-\lambda)\,\lambda - 12) - 2(-2\lambda - 6) - 3(-4 + 1 - \lambda) = 0$

$\Rightarrow \quad (-2-\lambda)\,(-\lambda + \lambda^2 - 12) - 2(-2\lambda - 6) - 3(-3 - \lambda) = 0$

$\Rightarrow \quad \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow \quad (\lambda + 3)\,(\lambda + 3)\,(\lambda - 5) = 0$

$\Rightarrow \quad \lambda = -3, -3, 5$

Thus, eigen values are $-3, -3, 5$.

To find eigen vector

for $\lambda = -3$, eigen vector is

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow \quad x + 2y - 3z = 0$

$\qquad 2x + 4y - 6z = 0$

$\qquad -x - 2y + 3z = 0$

From the above set, it can be seen that, $x + 2y - 3z = 0$ is only one independent equation.

So, let $z = 0$, we get $\quad x + 2y = 0$

$\Rightarrow \quad x = -2y$

$$\frac{x}{2} = \frac{y}{-1}, z = 0$$

Hence eigen vector is $(2, -1, 0)$ for $\lambda = -3$.

Also, to find another eigen vector for $\lambda = -3$.

Let $y = 0$, we get $\quad x - 3z = 0$

$$\Rightarrow \quad x = 3z$$

$$\Rightarrow \quad \frac{x}{3} = \frac{z}{1}$$

$$\Rightarrow \quad y = 0$$

So (3, 0, 1) is another eigen vector for $\lambda = -3$.

For $\lambda = 5$, eigen vector is

$$[A - \lambda I]X = 0$$

$$\Rightarrow \quad \begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x + 2y - 3z = 0 \qquad \text{...(1)}$$

$$\Rightarrow \quad 2x - 4y - 6z = 0 \qquad \text{...(2)}$$

$$-x - 2y - 5z = 0 \qquad \text{...(3)}$$

Multiply eqn. (1) by 2 and adding in eqn. (2), we have

$$-12x - 12z = 0$$

or $\quad -12x = 12z$

$$\Rightarrow \quad x = -z$$

Multiply eqn. (3) by 2 and adding in eqn. (2), we have

$$y = -2z$$

Hence $\quad \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$

So, (1, 2, –1) is eigen vector, corresponding to $\lambda = 5$.

Since all eigen vectors are L.I. and form basis of R^3.

$\therefore$ Eigen basis is {(2, –1, 0), (3, 0, 1) (1, 2, –1)}.

Example 5.72. *Find all the eigen values, eigen vectors and eigen basis of* $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Solution. Let $\quad A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

and $\quad |A - \lambda I| = 0$ (characteristic equation)

$$\Rightarrow \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (6 - \lambda)\,((3 - \lambda)^2 - 1) + 2(-2(3 - \lambda) + 2) + 2(2 - 2(3 - \lambda)) = 0$$

$$\Rightarrow \quad (6 - \lambda)\,(\lambda^2 - 6\lambda + 8) + 2(2\lambda - 4) + 2(2\lambda - 4) = 0$$

$$\Rightarrow \quad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\Rightarrow \qquad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$
$$\Rightarrow \qquad (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$
$$\Rightarrow \qquad (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$
$$\Rightarrow \qquad \lambda = 2, 2, 8$$

Eigen values are 2, 2, 8

For $\lambda = 2$, eigen vector is

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x - 2y + 2z = 0$$
$$\Rightarrow \qquad -2x + y - z = 0$$
$$2x - y + z = 0$$

Here $2x - y + z = 0$ is the only one independent equation

So, let $z = 0$

$$\Rightarrow \qquad 2x - y = 0$$

or $$2x = y$$

Thus, $$\frac{x}{1} = \frac{y}{2}, z = 0$$

Hence (1, 2, 0) is eigen vector for $\lambda = 2$.

Again for, $\lambda = 2$ to find another eigen vector,

Let us take, $y = 0$

Here $2x - y + z = 0$ becomes $2x + z = 0$

$$\Rightarrow \qquad z = -2x$$

or $$\frac{x}{1} = \frac{z}{-2}, y = 0$$

Hence (1, 0, –2) is eigen vector.

for, $\lambda = 8$, eigen vector is,

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad -2x - 2y + 2z = 0 \qquad \text{...(1)}$$
$$\Rightarrow \qquad -2x - 5y - z = 0 \qquad \text{...(2)}$$
$$2x - y - 5z = 0 \qquad \text{...(3)}$$

Multiply eqn. (2) by 2 and adding in eqn. (1), we have

$$-6x - 12y = 0$$
$$\Rightarrow \qquad x = -2y$$

Adding (2) and (3), we have

$$y = -z$$

$$\Rightarrow \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

(2, –1, 1) is eigen vector, corresponding to $\lambda = 8$.
After that since all eigen vectors are L.I. Hence form basis of R^3.
$\therefore$ Eigen basis is {(1, 2, 0), (1, 0, –2), (2, –1, 1)}.

Example 5.73. *Find all the eigen values, eigen vectors and eigen basis of* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\,((2-\lambda)(3-\lambda)-2) - 0 - 1(2 - 4 + 2\lambda) = 0$$

$$\text{or} \quad \lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda - 2\lambda + 2 = 0$$

$$\text{or} \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\text{or} \quad (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\text{or} \quad (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Eigen values are 1, 2, 3.
For $\lambda = 1$, eigen vector is $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 = 0 \qquad ...(1)$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0 \qquad ...(2)$$

$$2x_1 + 2x_2 + 2x_3 = 0 \qquad ...(3)$$

Putting eqn. (1) in (2) and (3), we have

$$x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2$$

$$\text{and} \quad 2x_1 + 2x_2 = 0$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

Thus eigen vector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for $\lambda = 1$.

For $\lambda = 2$, eigen vector is, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_3 = 0 \quad \Rightarrow \quad x_1 = -x_3$$

$\Rightarrow$
$$x_1 + x_3 = 0$$

$$2x_1 + 2x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 + 2x_2 = 0$$

$\Rightarrow$
$$\frac{x_1}{2} = \frac{x_2}{-1}\ ;\ x_3 = -x_1$$

Eigen vector is $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

For $\lambda = 3$, eigen vector is, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 0.x_2 - x_3 = 0 \quad \Rightarrow \quad -2x_1 = x_3 \qquad ...(1)$$

$\Rightarrow$
$$x_1 - x_2 + x_3 = 0 \qquad ...(3)$$

$$2x_1 + 2x_2 + 0x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 \qquad ...(2)$$

Put $x_1 = 1$, then, $\qquad x_2 = -1$

and $\qquad x_3 = -2(1) = -2 \qquad$ (from eqn. (1))

Eigen vector is $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is $\{(1, -1, 0), (2, -1, -2), (1, -1, -2)\}$.

Example 5.74. *Find all the eigen values, eigen vectors and eigen basis of* $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution. Characteristic equation is, $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (-5-\lambda)(-2-\lambda) - 4 = 0$

or $\quad \lambda^2 + 7\lambda + 10 - 4 = 0$

$\Rightarrow \quad \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow \quad \lambda = -6, -1$

$\therefore$ Eigen values are –6 and –1

For $\lambda = -1$, eigen vectors is, $[A - \lambda I]X = 0$

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$

Since $2x_1 - x_2$ is the only independent equation

Now, $\quad 2x_1 - x_2 = 0$

$\Rightarrow \quad 2x_1 = x_2$

or $\quad \dfrac{x_1}{1} = \dfrac{x_2}{2}$

Eigen vector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

for $\lambda = -6$, eigen vector is given by $[A - \lambda I]X = 0$

$$\begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0$

Since $x_1 + 2x_2$ is the only independent equation.

Now, $\quad x_1 + 2x_2 = 0$

$\Rightarrow \quad x_1 = -2x_2$

or $\quad \dfrac{x_1}{2} = \dfrac{x_2}{-1}$

Eigen vector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is $\{(1, 2), (2, -1)\}$.

Example 5.75. *Find all the eigen values, eigen vectors and eigen basis of matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$(3-\lambda)((2-\lambda)(5-\lambda) - 0) - 1(0) + 4(0) = 0$$

or $$(3-\lambda)(\lambda-5)(\lambda-2) = 0$$

$$\lambda = 2, 3, 5$$

Eigen values are 2, 3, 5 for eigen vectors.

For $\lambda = 2$, eigen vectors are $[A - \lambda I]X = 0$

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $$x_1 + x_2 + 4x_3 = 0 \qquad \text{...(1)}$$

$$6x_3 = 0$$

$$3x_3 = 0 \quad \Rightarrow \quad x_3 = 0 \qquad \text{...(2)}$$

Putting (2) in (1), $$x_1 + x_2 = 0$$

or $$\frac{x_1}{1} = \frac{-x_2}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

Eigen vector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

for $\lambda = 3$, eigen vector is, $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$0x_1 + x_2 + 4x_3 = 0$$

$$-x_2 + 6x_3 = 0$$

$$2x_3 = 0 \quad \Rightarrow \quad x_3 = 0$$

Thus, we have $\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{0}$

Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 5$, we have eigen vector is,

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$2x_1 - x_2 - 4x_3 = 0$$

$$3x_2 - 6x_3 = 0 \quad \Rightarrow \quad x_2 = 2x_3$$

$$2x_1 - 2x_3 - 4x_3 = 0 \qquad \text{[putting } x_2 = 2x_3\text{]}$$

or $\quad 2x_1 - 6x_3 = 0$

$\Rightarrow \quad x_1 = 3x_3$

$\Rightarrow \quad \dfrac{x_1}{3} = \dfrac{x_3}{1} \qquad ...(1)$

Also, $\quad x_2 = 2x_3$

$\therefore \quad \dfrac{x_2}{2} = \dfrac{x_3}{1} \qquad ...(2)$

From (1) and (2), $\dfrac{x_1}{3} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$, so the eigen vector is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is $\{(1, -1, 0), (1, 0, 0), (3, 2, 1)\}$.

Example 5.76. *Find all the eigen values, eigen vectors and eigen basis of* $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (2-\lambda)(-1-\lambda) - 10 = 0$

or $\quad \lambda^2 - \lambda - 12 = 0$

$\Rightarrow \quad (\lambda + 3)(\lambda - 4) = 0$

$\Rightarrow \quad \lambda + 3 = 0, \quad \lambda - 4 = 0$

$\quad \lambda = -3, \quad \lambda = 4$

Eigen values are -3 and 4.

Eigen vector for $\lambda = -3$

$$\begin{bmatrix} 2+3 & 2 \\ 5 & -1+3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow$ $5x + 2y = 0$

and $5x + 2y = 0$

Since there is only 1 independent equation

Thus, we have $5x + 2y = 0$

$\Rightarrow$ $5x = -2y$

$\Rightarrow$ $\dfrac{x}{-2} = \dfrac{y}{5}$

So eigen vector is $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$

For $\lambda = 4$, eigen vector is

$$\begin{bmatrix} 2-4 & 2 \\ 5 & -1-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-2x + 2y = 0$$

$$5x - 5y = 0$$

Again both are linearly dependent, we have

$$x - y = 0$$

$\Rightarrow$ $x = y$

or $\dfrac{x}{1} = \dfrac{y}{1}$

So eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since all the eigen vectors are L.I.

$\therefore$ Eigen basis is $\{(-2, 5), (1, 1)\}$.

Example 5.77. *Find all the eigen values, eigen vectors and eigen bases of* $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$.

Solution. (Students can try this) Eigen values are 1, –2. –2. Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. No eigen bases.

Example 5.78. *Find all the eigen values, eigen vectors and eigen bases of* $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution. Eigen values are 1, 1, 5. Eigen vectors are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Eigen basis is {(–2, 1, 0), (1, 0, –1), (1, 1, 1)}.

Example 5.79. *Find all the eigen values of* $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. *Hence find the eigen values of* A^{25}, *and* $A + 2I$.

Solution. Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda - 3 = 0$$

or $$(\lambda + 1)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = -1, \lambda = 3.$$

Eigen values of A^{25}

i.e., $$(-1)^{25} = -1$$

and $$(3)^{25} = 3^{25}$$

Eigen value of $A + 2I$

For $$A = 3, A + 2I = 3 + 2 = 5$$

For $$A = (-1), A + 2I = -1 + 2 = 1$$

Complex Eigen Values

Example 5.80. *Show that if* $0 < \theta < \pi$, *then* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *has no real eigen values and consequently no eigen vector.*

Solution. The characteristic equation is, $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2i\sqrt{1 - \cos^2\theta}}{2}$$

$$= \cos\theta \pm i \sin\theta$$

Hence the matrix A has no real eigen values and consequently no eigen vector.

Example 5.81. *For a given matrix* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *Find all the eigen values and eigen vectors of A. Is there an eigen basis for A?*

Solution. Here $\qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Characteristic equation is

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)^2 = 0$$

$$= \lambda = 1, 1 \text{ (Eigen values)}$$

To find eigen vector corresponding to eigen value $\lambda = 1$,

$$AX = \lambda X$$

$$[A - \lambda I]X = 0$$

$$[A - I]X = 0, (\lambda = 1)$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[If $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the corresponding eigen vector]

$\Rightarrow \qquad y = 0$

Take $x = 1$, then $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 1$.

Here, we can see that X does not form a basis of R^2, so we do not have an eigen basis for the given matrix A.

EXERCISE 5.11

Find the characteristic roots of the following matrices:

1. $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 3 & 11 \\ 0 & 3 & 17 \\ 0 & 0 & -2 \end{bmatrix}$ **3.** $\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$

Find the eigen values and corresponding eigen vectors for the following matrices:

4. $\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 5 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

Answers

1. $-1, 1, 2$ **2.** $2, 3, -2$ **3.** a, b, c

4. $0, (-3, 1, 0); 1, (12, -4, -1); 0, (-3, 1, 0)$ **5.** $1, (1, 1, -1); 2, (2, 1, 0)$

6. $2; (1, 0, 0)$ **7.** $1; (0, 0, 1)$ **8.** $-1(-3, -1, 3); 2(0, 1, 0); 3(1, 1, 1)$

9. $2(1, 0, 0); -2(0, 1, 1); -4(0, 1, -1)$

5.15. CAYLEY HAMILTON THEOREM

Definition: Every square matrix satisfies its characteristic equation *i.e.*, if the characteristic equation of the nth order of square matrix A is $|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots\ldots + k_n = 0$ ...(1)

then $$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots + k_n I = 0$$

General proof:

Let $$P = \text{adj }(A - \lambda I)$$

Since, the elements of $(A - \lambda I)$ are at the most of first degree in λ, the elements of $P = \text{adj }(A - \lambda I)$ are polynomials in λ of degree $(n - 1)$ or less.

We can therefore, split up P into a number of matrices each containing the same power of λ and write

$$P = P_1\lambda^{n-1} + P_2\lambda^{n-2} + P_{n-1}\lambda + P_n \qquad \text{...(2)}$$

Also, we know that if M is a square matrix, then M (adj M) $= |M| \times I$

$$\therefore \qquad (A - \lambda I)\, P = |A - \lambda I| \times I$$

By (1) and (2), we have

$$(A - \lambda I)\,(P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots\ldots + P_{n-1}\lambda + P_n)$$
$$= [(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2 \lambda^{n-2} + \ldots\ldots + k_{n-1} \lambda + k_n])\, I$$

Equating the co-efficients of like power of λ on both sides, we get

$$\begin{aligned} -P_1 &= (-1)^n I \qquad (\because \quad IP_1 = P_1) \\ AP_1 - P_2 &= k_1 I \\ AP_2 - P_3 &= k_2 I \\ &\vdots \\ &\vdots \\ AP_{n-1} - P_n &= k_{n-1} I \\ AP_n &= k_n I \end{aligned}$$

Pre-Multiplying these equation by $A^n, A^{n-1}, A^{n-2}, \ldots\ldots A, I$ respectively and adding, we get

$$0 = (-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots + k_{n-1} A + k_n I$$

or $$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots + k_{n-1} A + k_n I = 0 \qquad \text{...(3)}$$

Hence proved.

Note 1. Multiplying (3) by A^{-1}, we have

$$(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots + k_{n-1} I + k_n A^{-1} = 0$$

$$\Rightarrow \qquad A^{-1} = \frac{-1}{k_n} [(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots + k_{n-1} I]$$

Thus, Cayley-Hamilton theorem gives another method for computing the inverse of a matrix. Since this method expresses the inverse of a matrix of order n in terms of $(n - 1)$ powers of A it is suitable for computing the inverse of large matrices.

Note 2. If m be a positive integer such that $m > n$, then multiplying (3) by A^{m-n}, we get

$$(-1)^n A^m + k_1 A^{m-1} + k_2 A^{m-2} + \dots\dots + k_{n-1} A^{m-n+1} + k_n A^{m-n} = 0.$$

Showing that only positive integral power A^n $(m > n)$ of A is linearly expressible in terms of those of lower degree).

SOME SOLVED EXAMPLES

Example 5.82. *Using Cayley-Hamilton theorem, find the inverse of* $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0$$

Computing $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}$

$$= -(1-\lambda)(1+\lambda) - 4$$

$$= -1 + \lambda^2 - 4$$

$\Rightarrow \quad \lambda^2 - 5 = 0$

By Cayley-Hamilton theorem, put A in place of λ, gives the characteristic equation.

To show $\quad A^2 - 5I = 0$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Hence $\quad A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 0$

$$A^2 - 5I = 0$$

Multiplying both side by A^{-1}

$$A - 5A^{-1} = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} - 5\,[A^{-1}] = 0$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$ **Answer.**

Example 5.83. *Using Cayley-Hamilton theorem, find the inverse of* $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.

Solution. Try yourself.

Example 5.84. *If* $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, *express* $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ *as a linear polynomial in A.*

Solution. Let $$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$= (1 - \lambda)(3 - \lambda) + 2 = 0$$

$$= 3 - 3\lambda - \lambda + \lambda^2 + 2 = 0$$

$$= \lambda^2 - 4\lambda + 5 = 0$$

By Cayley-Hamilton theorem, we have

$$A^2 - 4A + 5I = 0 \qquad \text{...(1)}$$

Now $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$$= A^6 - 4A^5 + 5A^4 + 3A^4 - 12A^3 + 14A^2$$

$$= A^4 (A^2 - 4A + 5I) + 3A^4 - 12A^3 + 14A^2$$

$$= 0 + 3A^4 - 12A^3 + 14A^2 \text{ (By (1)}$$

$$= 3A^4 - 12A^3 + 15A^2 - A^2$$

$$= 3A^2 (A^2 - 4A + 5I) - A^2$$

$$= 0 - A^2$$

$$= 5I - 4A \qquad \text{By (I)}$$

Hence $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = 5I - 4A$. **Answer.**

Example 5.85. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ *Hence, compute* A^{-1}.

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e., $$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$= (2 - \lambda)[(2 - \lambda)(2 - \lambda) - 1] + 1[(-2 + \lambda) + 1] + 1[1 - 2 + \lambda]$$

$$= \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley-Hamilton theorem, we have to show that $A^3 - 6A^2 + 9A - 4I = 0$

Now, $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore$ $$A^3 - 6A^2 + 9A - 4I \qquad \text{...(1)}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies Cayley-Hamilton theorem.

Now, multiplying both sides of (1) by A^{-1}

we have $\quad A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4I\,A^{-1} = 0$

$$A^2 - 6A + 9I = 4A^{-1}$$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

or $$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ **Answer.**

Example 5.86. *Using Cayley-Hamilton theorem. Find* A^8 *if* $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e., $$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$-(1 - \lambda^2) - 4 = 0$$

or $\lambda^2 - 5 = 0$

or $\lambda^2 = 5$ ...(1)

By Cayley-Hamilton theorem, A satisfies its characteristic equation ...(1)

$\therefore$ $A^2 = 5I$

$= (A^2)^4 = (5I)^4$

$\Rightarrow$ $A^8 = 625\, I^4$

But, $I^4 = I$

So, $A^8 = 625\, I$ **Answer**

Example 5.87. *Using Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. *Find* A^{-1}.

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e., $\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(4 - \lambda)\,[1 + \lambda^2 - 2\lambda + 4] - 3\,[2 - 2\lambda + 2] + 1[4 - 1 + \lambda] = 0$

or $(4 - \lambda)\,(\lambda^2 - 2\lambda + 5) + 6\lambda - 12 + \lambda + 3 = 0$

or $4\lambda^2 - 8\lambda + 20 - \lambda^3 + 2\lambda^2 - 5\lambda + 7\lambda - 9 = 0$

or $-\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0$

or $\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$

Hence by Cayley-Hamilton theorem, we get characteristic equation

$$A^3 - 6A^2 + 6A - 11 = 0$$

Multiplying the equation by A^{-1}

$$A^2 - 6A + 6I - 11A^{-1} = 0$$

or $11A^{-1} = A^2 - 6A + 6I$

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6+1 & 12+3+2 & 4-6+1 \\ 8+2-2 & 6+1-4 & 2-2-2 \\ 4+4+1 & 3+2+2 & 1-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$$11\,A^{-1} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 23-24+6 & 17-18+0 & -1-6+0 \\ 8-12+0 & 3-6+6 & -2+12+0 \\ 9-6+0 & 7-12+0 & -2-6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix} \quad \textbf{Answer.}$$

Example 5.88. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and, hence, find the matrix represented by* $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

or $\quad (2-\lambda)\,[(1-\lambda)\,(2-\lambda)] + 1[\lambda - 1] = 0$

$\Rightarrow \quad (2-\lambda)^2\,(1-\lambda) + (\lambda - 1) = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \text{...(1)}$$

Now $\quad A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5\,(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= 0 + 0 + A^2 + A + I \qquad \text{[By (1)]}$$

$$= A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\Rightarrow \quad A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix} \quad \textbf{Answer.}$$

Example 5.89. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$. *Hence compute* A^{-1}.

Solution. Try yourself.

Answer: $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

Example 5.90. *Using Cayley-Hamilton theorem, find* A^6 *if* $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$.

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 1 \\ 5 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad [-(2-\lambda)(2+\lambda) - 5] = 0$$

$$\Rightarrow \quad [-(2^2 - \lambda^2) - 5] = 0$$

$$\Rightarrow \quad [-4 + \lambda^2 - 5] = 0$$

$$\Rightarrow \quad \lambda^2 - 9 = 0$$

$$\Rightarrow \quad \lambda^2 = 9 \qquad \text{...(1)}$$

By Cayley Hamilton theorem A satisfies the eq. (1)

$$A^2 - 9I = 0$$

$$\Rightarrow \quad A^2 = 9I$$

$$\Rightarrow \quad (A^2)^3 = (9I)^3$$

$$\Rightarrow \quad A^6 = 729\, I \quad \textbf{Answer.}$$

Example 5.91. *Show that the matrix* $A = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$ *satisfies Cayley-Hamilton theorem.*

Solution.
$$A = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & r & -q \\ -r & 0-\lambda & p \\ q & -p & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda(\lambda^2 + p^2) - r(r\lambda - pq) - q(rp + q\lambda) = 0$$

$$\Rightarrow \quad -\lambda^3 - \lambda p^2 - r^2\lambda + r\,pq - rpq - q^2\lambda = 0$$

$$\Rightarrow \quad -\lambda^3 - \lambda p^2 - r^2\lambda - q^2\lambda = 0$$

$$\Rightarrow \quad \lambda^3 + \lambda p^2 + r^2\lambda + q^2\lambda = 0 \qquad \text{...(1)}$$

By Cayley-Hamilton theorem.

$$A^3 + Ap^2 + r^2A + q^2A = 0 \qquad \text{...(2)}$$

$$A^2 = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}\begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -r^2 - q^2 & qp & rp \\ pq & -r^2 - p^2 & rq \\ pr & qr & -q^2 - p^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -r^2 - q^2 & qp & rp \\ pq & -r^2 - p^2 & rq \\ pr & qr & -q^2 - p^2 \end{bmatrix}\begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -pqr + pqr & -r^3 - rq^2 - rp^2 & r^2q + q^3 + qp^2 \\ r^3 + rp^2 + rq^2 & rpq - pqr & -pq^2 - r^2p - p^3 \\ -qr^2 - q^3 - qp^2 & pr^2 + pq^2 + p^3 & -pqr + pqr \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -r(r^2 + q^2 + p^2) & q(r^2 + q^2 + p^2) \\ r(r^2 + p^2 + q^2) & 0 & -p(r^2 + q^2 + p^2) \\ -q(r^2 + p^2 + q^2) & p(r^2 + q^2 + p^2) & 0 \end{bmatrix}$$

$$= (r^2 + p^2 + q^2)\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

$$= -(r^2 + p^2 + q^2)\begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$$

$$= -(r^2 + p^2 + q^2)\, A$$

From the eqn. (1), we have

$$A^3 + Ap^2 + r^2A + q^2A = -(r^2 + p^2 + q^2)\, A + (p^2 + r^2 + q^2)\, A$$

$= 0$ Hence it satisfies Cayley Hamilton theorem.

Example 5.92. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and use it to find the matrix represented by* $A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$.

Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as a linear polynomial in A.*

Solution. The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - \lambda)(3 - \lambda) - 8 = 0$

$\Rightarrow \quad 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

or $\quad \lambda^2 - 4\lambda - 5 = 0$

By Cayley-Hamilton theorem, we have

$$A^2 - 4A - 5I = 0 \qquad \text{...(1)}$$

Now, $\quad A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$

$= A^3 [A^2 + 5A - 6I] + 2(A^2 - 2A) + 7I \qquad$ (By (1))

$= A^3 [4A + 5I + 5A - 6I] + 2(4A + 5I - 2A) + 7I$

$= A^3 [9A - I] + 2[2A + 5I] + 7I$

$= A^2 (9A^2 - A) + 4A + 10I + 7I$

$= (4A + 5I)(36A + 45I - A) + 4A + 17I$

$= (4A + 5I)(35A + 45I) + 4A + 17I$

$= 5(4A + 5I)(7A + 9I) + 4A + 17I$

$= 5(28A^2 + 36A + 35A + 45I) + 4A + 17I$

$= 5(28A^2 + 71A + 45I) + 4A + 17I$

$= 5(112A + 140I + 71A + 45I) + 4A + 17I$

$= 5(183A + 185I) + 4A + 17I$

$= 919A + 942I$

$$= 919 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 942 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 919 & 3676 \\ 1838 & 2757 \end{bmatrix} + \begin{bmatrix} 942 & 0 \\ 0 & 942 \end{bmatrix}$$

$$= \begin{bmatrix} 1861 & 3676 \\ 1838 & 3699 \end{bmatrix}$$ **Answer.**

Now, $\quad A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= A^5 - 4A^4 - 5A^3 - 2A^3 + 11A^2 - A - 10I$

$= A^3 [A^2 - 4A - 5I] - 8A^2 - 10A + 11A^2 - A - 10I$

$= 0 + 3A^2 - 11A - 10I$

$= 12A + 15I - 11A - 10I$

$= A + 5I$ **Answer.**

Example 5.93. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ *and hence obtain the inverse of given matrix.*

Solution. Try yourself.

Answer: $A^{-1} = \dfrac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$

EXERCISE 5.12

Verify Cayley Hamilton theorem for the following matrices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 2 & 0 & 3 \end{bmatrix}$

Verify Cayley Hamilton theorem for the following matrices and compute A^{-1} also:

3. $\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

5. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ and show that A satisfies this equation.

6. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, use Cayley-Hamilton theorem to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Answers

3. $\begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix}$

5. $\lambda^3 = 0$

6. $138A - 403I$

5.16 DIAGONALIZATION OF MATRICES

Let $x_1, x_2, ..., x_n$ be the eigen vectors of a square matrix A, corresponding to the eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, respectively, then

$$Ax_i = \lambda_i x_i,$$

Denote by 'B' the square matrix whose columns are $x_1, x_2, ..., x_n$. For bravity, we shall write P as $[x_1, x_2, ..., x_n]$, then

$$\begin{aligned} AP &= A[x_1\ x_2 \ldots x_n] \\ &= [Ax_1\ Ax_2 \ldots Ax_n] \\ &= [\lambda_1 x_1\ \lambda_2 x_2 \ldots \lambda_n x_n] \\ &= PD \end{aligned}$$

[This step can be verified by writing P and D as full and multiplying] where $$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

This gives $P^{-1} AP = D$

This method fails when 'n' linearly independent eigen vectors do not exist for A. Such matrices are not diagonalizable.

When n linearly independent eigen vectors exist, P is non-singular and $P^{-1} AP$ is a diagonal matrix with eigen values as its diagonal elements.

The matrix P which is used to diagonalize the matrix A is called the modal matrix of A and the diagonal matrix thus obtained is known as spectral matrix of A.

Two matrices A and C are said to be similar if there is a non-singular matrix B such that

$$C = B^{-1} AB$$

Evidently, a diagonalizable matrix A is similar to a diagonal matrix D.

Note: Similar matrices have the same eigen values.

INTERESTING FACTS

1. It simplifies significantly certain computations. However, the first use of diagonalization can be seen in **Markov processes**, where the power of some square matrix are used extensively and Markov processes are really rich in applications, such as
   - Market
   - Weather forecasting
   - Genetics
   - Diffusion of gasses
   - The most famous one is probably Google's page ranking algorithm.
2. It is also use in Mechanics, for example, a way to find principal axes of inertia (with tensor of inertia being the diagonalized matrix).
3. One other thing is finding normal modes of an oscillating system (which requires simultaneous diagonalization of two matrices of kinetic and potential energy

VIDEO REFERENCES

Eigenvalues & Eigenvectors

Diagonaliz-ation of Matrices

APPLICATIONS TO REAL LIFE

- One important use of diagonalisation is for computing higher powers of matrix efficiently.

  If $A = M^{-1} DM$, then $A^n = M^{-1} D^n M$

  The above property makes it easy to compute higher powers of matrix A, since computing D^n is much more easy as compared with computing A^n.

SOME SOLVED EXAMPLES

Example 5.94. *Diagonalize* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ *and obtain the modal matrix.*

Solution. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1-\lambda)[-\lambda(2-\lambda)+1] - 2[-\lambda+1] - 2[-1+2-\lambda] = 0$$

$$\Rightarrow \quad (-1-\lambda)[\lambda^2 - 2\lambda + 1] - 2[-\lambda+1] - 2[-\lambda+1] = 0$$

$$\Rightarrow \quad (-1-\lambda)(1-\lambda)^2 - 4(1-\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)[(-1-\lambda)(1-\lambda) - 4] = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2 - 5) = 0$$

$$\Rightarrow \quad \lambda = 1, \pm\sqrt{5}$$

Now, we find the eigenvectors corresponding to these eigen values.

For $\lambda = 1$, let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be an eigen vector such that

$$AX_1 = \lambda X_1$$

$$\Rightarrow \quad (A - \lambda I)X_1 = 0$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 2y_1 - 2z_1 = 0 \quad \Rightarrow \quad -x_1 + y_1 - z_1 = 0 \qquad \text{...(1)}$$

$$x_1 + y_1 + z_1 = 0 \qquad \text{...(2)}$$

$$-x_1 - y_1 - z_1 = 0 \quad \Rightarrow \quad x_1 + y_1 + z_1 = 0$$

(1) + (2) $\Rightarrow$ $\quad 2y_1 = 0 \quad \Rightarrow \quad y_1 = 0$

From (2) $\quad -x_1 - z_1 = 0 \quad \Rightarrow \quad -x_1 = z_1$

Take $\quad z_1 = -1, \quad \Rightarrow \quad x_1 = 1$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda = \sqrt{5}$, let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be an eigenvector such that

$$[A-\sqrt{5}I]\ X_2 = 0$$

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & 0-\sqrt{5} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1-\sqrt{5})\,x_2 + 2y_2 - 2z_2 = 0 \qquad ...(4)$$

$$x_2 + (2-\sqrt{5})\,y_2 + z_2 = 0 \qquad ...(5)$$

$$-x_2 - y_2 - \sqrt{5}\,z_2 = 0 \qquad ...(6)$$

(5) + (6) $\Rightarrow$ $(1-\sqrt{5})y_1 + (1-\sqrt{5})z_1 = 0$

$\Rightarrow$ $y_1 = -z_1$

Take $z_1 = 1$, then $y_1 = -1$

From (6) $-x_2 = y_2 + \sqrt{5}z_2 = -1 + \sqrt{5}$

$\Rightarrow$ $x_2 = 1 - \sqrt{5}$

$\therefore$ $X_2 = \begin{bmatrix} 1-\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$

For $\lambda = -\sqrt{5}$, let $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be an eigenvector such that

$$\begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & 0+\sqrt{5} \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1+\sqrt{5})\,x_3 + 2y_3 - 2z_3 = 0 \qquad ...(7)$$

$$x_3 + (2+\sqrt{5})\,y_3 + z_3 = 0 \qquad ...(8)$$

$$-x_3 - y_3 + \sqrt{5}\,z_3 = 0 \qquad ...(9)$$

(8) + (9) $\Rightarrow$ $(1+\sqrt{5})\,y_3 + (1+\sqrt{5})\,z_3 = 0$

$\Rightarrow$ $y_3 = -z_3$

Take $z_3 = 1, y_3 = -1$

From (9) $-x_3 = y_3 - \sqrt{5}\ z_3$

$\Rightarrow$ $-x_3 = -1 - \sqrt{5}$

$$\therefore \qquad X_3 = \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \qquad \text{Modal matrix } P = \begin{bmatrix} 1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|P| = -2\sqrt{5}$$

$$P^{-1} = \frac{1}{(-2\sqrt{5})}\begin{bmatrix} 0 & 1 & -1 \\ 2\sqrt{5} & 2+\sqrt{5} & -2+\sqrt{5} \\ 2\sqrt{5} & 1 & -1 \end{bmatrix}^T = \frac{1}{(-2\sqrt{5})}\begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -2+\sqrt{5} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ \frac{-1}{2\sqrt{5}} & \frac{-1}{\sqrt{5}}-\frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{5}}-\frac{1}{2} & \frac{1}{2\sqrt{5}} \end{bmatrix}$$

$$P^{-1}AP = D$$

$$\Rightarrow \qquad D = \begin{bmatrix} 0 & -1 & -1 \\ \frac{-1}{2\sqrt{5}} & \frac{-1}{\sqrt{5}}-\frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{5}}-\frac{1}{2} & \frac{1}{2\sqrt{5}} \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ \frac{-1}{2} & \frac{-5-2\sqrt{5}}{2\sqrt{5}} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5-2\sqrt{5}}{2\sqrt{5}} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}.$$

Example 5.95. *Show that the matrix* $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ *is not diagonalizable over the field C.*

Solution. Given $\qquad A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Corresponding characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \qquad \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad \lambda = 2, 2$$

Thus the only distinct eigen value is 2

If $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the corresponding eigen vector, then

$$AX = \lambda X$$

or $\qquad [A - \lambda I]X = 0$

For $\lambda = 2$, $\qquad [A - 2I]X = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad 0x + y = 0 \quad \Rightarrow \quad y = 0$$

Taking $x = 1, y = 0, X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the only eigen vector corresponding to $\lambda = 2$.

Thus the given square matrix A has only one linearly independent eigen vector.

So the given square matrix A is not diagonalizable.

[For the given matrix A to be diagonalizable, it must have 2 linearly independent eigen vectors]

Example 5.96. *Show that the matrix* $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ *is diagonalizable.*

Solution. First we will find the eigen values and eigen vectors in the same manner as we have done in Example 5.72 on page no. 5.96.

After that, to check for diagonalizability.

Since, the given matrix A has 3 linearly independent eigen vectors.

So the given matrix is diagonalizable.

Example 5.97. *Check the diagonalizability of the given matrix* $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution. Proceeding in a similar way as in Example 5.73 on page no. 5.98.

Then, to check for diagonalization

Since the given matrix A has 3 linearly independent eigen vectors.

So the given matrix is diagonalizable,

EXERCISE 5.13

1. Show that the given matrices are diagonalizable:

i. $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ ii. $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

2. Show that the given matrix $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$ is not diagonalizable.

3. Check the diagonalizability of $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$.

4. Whether the given matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable or not? Support your answer by giving proper reason.

Answers

3. Diagonalizable

4. Yes, Diagonalisable, as A.M. of each eigen value = G.M. of each eigen value.

5.17 QUADRATIC FORMS

Definition: A homogeneous polynomial of the type

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j$$

where a_{ij} are elements of a field F is called a *quadratic form* in n variables $x_1, x_2, ..., x_n$ over the field F. If a_{ij} are real, then the quadratic form is called real *quadratic form*.

For example, $x_1^2 - 3x_1x_2 + x_2^2 + x_1x_3$ is a real quadratic form.

Theorem 1. *Every quadratic form over a field F in n variables $x_1, x_2, ..., x_n$ can be expressed in the form of $X^T BX$, where B is a symmetric matrix of order n over F and X is a column vector $[x_1, x_2, ..., x_n]^T$.*

Proof. Let $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j$,

be a quadratic form over the field F in n variables $x_1, x_2, ..., x_n$. Since x_i, x_j are scalars, we have $x_i\, x_j = x_j\, x_i$. Therefore, the coefficient of $x_i\, x_j$ is $a_{ij} + a_{ji}$. Thus, we assign half of the coefficient to x_{ij} and half to x_{ji}. Let b_{ij} be another set of scalars such that $b_{ii} = a_{ii}$ and $b_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ for $i \neq j$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j.$$

Since, $b_{ij} = b_{ji}$, the matrix $B = [b_{ij}]_{n\times n}$ is symmetric. We further note that if

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix},$$

then
$$X^T BX = [x_1\ x_2 \dots x_n] \begin{bmatrix} b_{11} & b_{12} & \dots & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j.$$

The symmetric matrix B is called the matrix of the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j.$$

SOME SOLVED EXAMPLES

Example 5.98. *Find the matrix of the quadratic form* $x_1^2 - 3x_1x_2 + x_2^2 + x_1x_3$.

Solution. The given quadratic form can be written as

$$x_1^2 - \frac{3}{2}x_1x_2 - \frac{3}{2}x_2x_1 + x_2^2 + \frac{1}{2}x_1x_3 + \frac{1}{2}x_3x_1..$$

Therefore, the matrix of the given quadratic form is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{11} = 1,\quad a_{12} = -\frac{3}{2}\quad a_{13} = \frac{1}{2}$

$a_{21} = -\frac{3}{2},\quad a_{22} = 1,\quad a_{23} = 0$

$a_{31} = \frac{1}{2},\quad a_{32} = 0,\quad a_{33} = 0.$

Hence $$A = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \text{ which is symmetric.}$$

Example 5.99. *Write down the matrix of the quadratic form*

$$x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3.$$

Solution. $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3$.

coeff. of $x_1^2 = 1 = a_{11}$, coeff. of $x_2^2 = 2 = a_{22}$, coeff. of $x_3^2 = -7 = a_{33}$

$$\frac{1}{2}\text{ coeff. of } x_1x_2 = \frac{1}{2}(-4) = a_{12}, \quad \frac{1}{2}\text{ coeff. of } x_1x_3 = \frac{1}{2}(8) = a_{13},$$

$$\frac{1}{2}\text{ coeff. of } x_2x_3 = \frac{1}{2}(5) = a_{23}$$

(1) can be expressed as $= X'AX$, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix}$$

Given quadratic form $= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **Answer**

Example 5.100. *Write down the quadratic form corresponding to the matrix* $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$.

Solution. Quadratic form $= X AX$

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 + 5x_3 & 2x_1 + 3x_3 & 5x_1 + 3x_2 + 4x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 2x_1x_2 + 5x_3x_1 + 2x_1x_2 + 3x_2x_3 + 5x_1x_3 + 3x_2x_3 + 4x_3^2$$

$$= x_1^2 + 4x_3^2 + 4x_1x_2 + 10x_1x_3 + 6x_2x_3$$ **Answer.**

5.17.1 Diagonalisation of Quadratic Forms

We know that for every real symmetric matrix A there exists an orthogonal matrix U such that

$$U^T AU = \text{diag}\,[\lambda_1\ \ \lambda_2 \ldots \lambda_n],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are characteristic roots of A.

Applying the orthogonal transformation $X = UY$ to the quadratic form $X^T AX$, we have

$$X^T AX = \lambda_1 y_1^{\,2} + \lambda_2 y_2^{\,2} + \ldots + \lambda_n y_n^{\,2}.$$

If the rank of A is r, then n-r characteristic roots are zero and so

$$X^T AX = \lambda_1 y_1^{\,2} + \lambda_2 y_2^{\,2} + \ldots + \lambda_r y_r^{\,2},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-zero characteristic roots.

Definition. A square matrix B of order n over a field F is said to be congruent to another square matrix A of order n over F, if there exists a non-singular matrix P over F such that $B = P^T AP$.

The relation of "congruence of matrices" is an equivalence relation in the set of all $n \times n$ matrices over a field F. Further, let A be symmetric matrix and let B be congruent to A. Therefore, there exists a non-singular matrix P such that $B = P^T AP$. Then

$$\begin{aligned} B^T &= (P^T AP)^T = P^T A^T P \\ &= P^T AP, \text{ since } A \text{ is symmetric} \\ &= B. \end{aligned}$$

Hence, every *matrix congruent to a symmetric matrix is a symmetric matrix.*

Theorem 1. (Congruent reduction of a symmetric matrix). If A is any n rowed non-zero symmetric matrix of rank r over a field F, then there exists and n rowed non-singular matrix P over F such that

$$P^T AP = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where A_1 is a non-zero singular diagonal matrix of order r over F and each **0** is a null matrix of a suitable size.

Corollary 1. Corresponding to every quadratic form $X^T AX$ over a field F, there exists a non-singular linear transformation $X = PY$ over F such that the form $X^T AX$ transforms to

$$\lambda_1 y_1^{\,2} + \lambda_2 y_2^{\,2} + \ldots + \lambda_r y_r^{\,2},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are scalars in F and r is the rank of the matrix A.

Definition. The rank of the symmetric matrix A is called the rank of the quadratic form $X^T AX$.

Corollary 2. If $X^T AX$ is a real quadratic form of rank r in n variables, then there exists a real non-singular linear transformation $X = PY$ which transform $X^T AX$ to the form

$$Y^T P^T APY = y_1^{\,2} + y_2^{\,2} + \ldots + y_p^{\,2} - y^2{}_{p+1} - \ldots - y_r^{\,2},$$

which is called *canonical form* or *normal form* of a real quadratic form.

The number of positive terms in the normal form of $X^T AX$ is called the *index* of the quadratic form, whereas $p - (r - p) = 2p - r$ is called the *signature* of the quadratic form and is usually denoted by s. A quadratic form $X^T AX$ with a non-singular matrix A of order n is called *positive definite* if $n = r = p$, that is, if n = rank = index. A quadratic form is called *positive semi-definite* if $r < n$ and $r = p$. Similarly a quadratic form is called *negative semi-definite* if $r < n$ and its index is zero.

SOME SOLVED EXAMPLES

Example 5.101. *Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into canonical form.*

Solution. The given quadratic form can be written as $X'AX$ where $X' = [x, y, z]$ and the symmetric matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$$

Let us reduce A into diagonal matrix. We know that $A = I_3 A I_3$

i.e.,
$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 - \frac{2}{3}R_1,\ R_3 \to R_3 - \frac{4}{3}R_1$ (for A on L.H.S. and pre-factor on R.H.S.), we get

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - \frac{2}{3}C_1,\ C_3 \to C_3 - \frac{4}{3}C_1$ (for A on L.H.S. and post-factor on R.H.S.), we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_2 + R_3$, we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + C_2$, we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Diag}\left(3, -\frac{4}{3}, -1\right) = P'AP$$

The canonical form of the given quadratic form is

$$Y'(P'AP)Y = [y_1, y_2, y_3]\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3{y_1}^2 - {y_3}^2$$

Hence $\rho(A) = 3$, index = 1, signature = 1 – 2 = –1.

Example 5.102. *Reduce the quadratic form* $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ *to the sum of squares and find the corresponding linear transformation. Also find the index and signature.*

Solution. The matrix of the given quadratic form Q is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$IAI = A$$

$$\underset{R}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A \underset{S}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Row transformation carried out on R.H.S. will be applied on R prefactor matrix. Column transformation applied on R.H.S. will be applied on S post-factor matrix.

$$R_2 \to R_2 + \frac{1}{3}R_1, R_3 \to R_3 - \frac{1}{3}R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ 0 & \frac{7}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$C_2 \to C_2 + \frac{1}{3}C_1,\ C_3 \to C_3 - \frac{1}{3}C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$R_3 \to R_3 + \frac{1}{7}R_2,\ C_3 \to C_3 + \frac{1}{2}C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{-2}{7} & \frac{1}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{bmatrix}$$

Thus the matrix A is reduced to the diagonal form B

$$P'AP = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{bmatrix}, \quad \text{where} \quad P = \begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

The canonical form (sum of the squares) is

$$Q = y'By = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2$$

$$X = PY \quad i.e. \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{-2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3, \quad x_2 = y_2 + \frac{1}{7}y_3, \quad x_3 = y_3$$

The rank of quadratic form $(r) = 3$

The index of quadratic form $(P) = 3$

The signature of quadratic form $r - (r - P) = 3$. **Answer**

5.18 REDUCTION OF QUADRATIC FORM TO CANONICAL FORM

A homogeneous expression of the second degree in any number of variables is called a *quadratic form*.

For instance, if $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X' = [x \quad y \quad z]$, then

$$X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \qquad \ldots(i)$$

which is a *quadratic form*.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of the matrix A and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be its corresponding eigen vectors in the normalized form (*i.e.* each element is divided by square root of sum of the squares of all the three elements in the eigen vector).

Then, $$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{where } P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Hence the *quadratic form (i) is reduced to a* **sum of squares** (*i.e.* **canonical form**)

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

and P is the **matrix of transformation** which is an orthogonal matrix. That is why the above method of reduction is called the *orthogonal transformation*.

Steps:

1. Convert quadratic form to matrix form.
2. Find eigen values and eigen vectors.
3. Find Modal Matrix (P).
4. Find Normalised Matrix (N).
5. Find N^T.
6. Find D ($D = N^T AN$).

SOME SOLVED EXAMPLES

Example 5.103. *Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form. Also specify the matrix of transformation.*

Solution. The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Its characteristic equation is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$

which gives $\lambda = 2, 3, 6$ as its eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 \quad i.e. \quad 2x^2 + 3y^2 + 6z^2.$$

To find the matrix of transformation

From $[A - \lambda I]X = 0$, we obtain the equations

$$(3-\lambda)x - y + z = 0;\ -x + (5-\lambda)y - z = 0;\ x - y + (3-\lambda)z = 0.$$

Now corresponding to $\lambda = 2$, we get

$$x - y + z = 0, -x + 3y - z = 0 \text{ and } x - y + z = 0,$$

whence $$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$\therefore$ The eigen vector is $(1, 0, -1)$ and its normalised form is $(1/\sqrt{2}, 0, -1/\sqrt{2})$.

Similarly corresponding to $\lambda = 3$, the eigen vector is $(1, 1, 1)$ and its normalised form is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Finally, corresponding to $\lambda = 6$, the eigen vector is $(1, -2, 1)$ and its normalised form is $(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})$.

Hence the matrix of transformation is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$.

Example 5.104. *Reduce $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ into canonical form by orthogonal transformation.*

Solution. The matrix of the quadratic form is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic roots of A are given by $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\therefore \qquad \lambda = 0, 3, 15$$

Characteristic vector for $\lambda = 0$ is given by $[A - (0)I]\, X = O$

i.e., $$\begin{aligned} 8x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 7x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

Solving first two, we get $\dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{2}$ giving the eigen vector $X_1 = k_1(1, 2, 2)'$.

When $\lambda = 3$, the corresponding characteristic vector is given by $[A - 3I]X = O$

i.e., $$\begin{aligned} 5x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 4x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 &= 0 \end{aligned}$$

Solving any two equations, we get

$$X_2 = k_2(2, 1, -2)'.$$

Similarly characteristic vector corresponding to $\lambda = 15$ is $X_3 = k_3(2, -2, 1)'$.

Now, X_1, X_2, X_3 are pairwise orthogonal *i.e.*, $X_1 . X_2 = X_2 . X_3 = X_3 . X_1 = 0$.

$\therefore$ The normalised modal matrix is

$$B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Now B is orthogonal matrix and $|B| = 1$

i.e., $$B^{-1} = B^T \text{ and } B^{-1}AB = D = \text{diag}(0, 3, 15)$$

i.e., $$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$X'AX = Y'(B^{-1}AB)Y = Y'DY$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 3y_2^2 + 15y_3^2$$

which is the required canonical form.

EXERCISE 5.14

1. Write the matrix for the given quadratic form

a. $Q = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

b. $Q = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

c. $Q = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

2. Reduce the quadratic form $2xy + 2yz + 2zx$ to canonical form.

3. Reduce the given quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 + 4x_2x_3$ into canonical form.

4. Let the rank of quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + cx_3^2 + 4x_1x_3 + 4x_2x_3$ be 2:

a. Find the parameter c;

b. Find an invertible transform which can be changed into canonical form;

c. What happens, if $f(x_1, x_2, x_3) = 1$.

Answers

1. a. $A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$ b. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

2. $2x^2 - y^2 - z^2$ **3.** $y_1^2 + y_2^2 + y_3^2$

4. a. $c = 9$ b. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

c. It will become an elliptic cylinder; $y_2^2 + 9y_3^2 = 1$.

SUBJECTIVE SOLVED QUESTIONS (HOTS)

Example 1. *Determine the number of values of k for which the system of equation*

$$(k+1)x + 8y = 4k$$
$$kx + (k+3)y = 3k-1$$

has infinitely many solution.

Solution. Given system of linear equation is

$$(k+1)x + 8y = 4k$$
$$kx + (k+3)y = 3k-1$$

It can be written in the form $AX = B$,

where $$A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$$

and $$B = \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$$

Now, $$[A : B] = \begin{bmatrix} k+1 & 8 & : & 4k \\ k & k+3 & : & 3k-1 \end{bmatrix}$$

Operating $$R_2 \to R_2 - \frac{k}{k+1} R_1$$

$$[A : B] \sim \begin{bmatrix} k+1 & 8 & : & 4k \\ 0 & (k+3) - \dfrac{8k}{k+1} & : & (3k-1) - \dfrac{4k^2}{k+1} \end{bmatrix}$$

$$\sim \begin{bmatrix} k+1 & 8 & : & 4k \\ 0 & \dfrac{k^2-4k+3}{k+1} & : & \dfrac{-k^2+2k-1}{k+1} \end{bmatrix}$$

It is given that system of equation has infinitely many solution

$\therefore$ $$\rho(A) = \rho(A : B) < n\ (= 2)$$

For this, $$\frac{k^2-4k+3}{k+1} = 0 \qquad ...(1)$$

and $$\frac{-k^2+2k-1}{k+1} = 0 \qquad ...(2)$$

From (1), $k^2 - 4k + 3 = 0$

$\Rightarrow$ $(k-3)(k-1) = 0$

$\Rightarrow$ $k = 3, 1$

From (2), $-k^2 + 2k - 1 = 0$

$\Rightarrow$ $k^2 - 2k + 1 = 0$

$\Rightarrow$ $(k-1)^2 = 0$

$\Rightarrow \qquad k = 1$

$\therefore$ $k = 1$ is the only solution for which system of equation has infinitely many solution.

Example 2. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $A^{-1} = \frac{1}{6}(A^2 + cA + dI)$ *where* $c, d \in R$, *then find the value of* (c, d).

Solution. We have $\quad |A| = 1(4 + 2) - 0 + 0 = 6$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Now,
$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

Also,
$$cA = \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix}$$

and
$$dI = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

Given that
$$A^{-1} = \frac{1}{6}(A^2 + cA + dI)$$

$$\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \frac{1}{6}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}\right)$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+c+d & 0 & 0 \\ 0 & -1+c+d & 5+c \\ 0 & -10-2c & 14+4c+d \end{bmatrix}$$

By equality of matrices, equating corresponding elements, we get

$$6 = 1 + c + d \quad \Rightarrow \quad 5 = c + d$$

$$-1 = 5 + c \quad \Rightarrow \quad -6 = c$$

So, $\quad 5 = -6 + d \quad \Rightarrow \quad d = 11$

So, $(-6, 11)$ is the required value.

Example 3. *Consider matrix* $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}$ *and vector* $X = [X_1\ X_2]^T$.

Find the number of distinct real values of k for which the equation AX = 0 has infinitely many solution.

Solution. The given system has infinitely many solution.

$$\therefore \qquad |A| = 0$$

or
$$\begin{vmatrix} k & 2k \\ k^2 - k & k^2 \end{vmatrix} = 0$$

i.e.,
$$k^3 - 2k(k^2 - k) = 0$$
$$k^3 - 2k^3 + 2k^2 = 0$$
$$-k^3 + 2k^2 = 0$$
$$k^2(-k + 2) = 0$$
$$k = 0 \quad \text{or} \quad k = 2$$

Hence, k has two values for which given system of linear equation has infinitely many solution.

Example 4. *The matrix* $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ *has det. A = 100 and trace A = 14. Find the value of* $|a - b|$.

Solution. Given that trace $(A) = 14$

$$\Rightarrow \qquad a + 5 + 2 + b = 14$$
$$a + 7 + b = 14$$
$$a + b = 7 \qquad \text{...(1)}$$

Also, $\qquad$ det. $A = 100$

Expanding about R_4,

$$b\begin{vmatrix} a & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 100$$

$$\Rightarrow \quad b[a(10 - 0) - 0 + 3(0)] = 100$$
$$\Rightarrow \qquad 10ab = 100$$
$$\Rightarrow \qquad ab = 10$$
$$b = \frac{10}{a} \qquad \text{...(2)}$$

Putting the value of b from (2) in (1), we get

$$a + \frac{10}{a} = 7$$
$$\Rightarrow \qquad a^2 + 10 = 7a$$
$$\Rightarrow \qquad a^2 - 7a + 10 = 0$$
$$\Rightarrow \qquad (a - 5)(a - 2) = 0$$

$\Rightarrow \qquad a = 5 \text{ or } 2$

From (2), $\qquad b = 2 \text{ or } 5$

Now, $\qquad |a - b| = |5 - 2| \text{ or } |2 - 5|$

$$= 3.$$

Example 5. *Consider 5 × 5 matrix* $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$.

It is given that A has only one real eigen value, then find that real eigen value of A.

Solution. Characteristic equation is

$$|A - \lambda I| = 0$$

$$\therefore \qquad \begin{vmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

Operating $R_1 \to R_1 + R_2 + R_3 + R_4 + R_5$

$$\begin{vmatrix} 15-\lambda & 15-\lambda & 15-\lambda & 15-\lambda & 15-\lambda \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

Now, taking common

$$(15 - \lambda) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$$(15 - \lambda) \,.\, |\text{Matrix}| = 0$$

$\Rightarrow \qquad 15 - \lambda = 0$

$\Rightarrow \qquad \lambda = 15$

$\therefore$ 15 is real eigen value of A.

Another approach

If sum of all rows or columns are same then that sum will be the eigen value of matrix.

In matrix A, Sum of all rows = 15

$\therefore$ 15 is a real eigen value of A.

Example 6. *Given that* $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *then what is the value of* A^3.

Solution. Given, $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & -3 \\ 2 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(-5 - \lambda)(-\lambda) + 6 = 0$

$\Rightarrow$ $5\lambda + \lambda^2 + 6 = 0$

$\Rightarrow$ $\lambda^2 + 5\lambda + 6 = 0$

Since, every matrix satisfies its characteristic equation (Cayley-Hamilton Theorem)

$\therefore$ $A^2 + 5A + 6I = 0$

$\Rightarrow$ $A^2 = -5A - 6I$ ...(1)

Multiply by A on both sides, we have,

$$A^3 = -5A^2 - 6AI$$

$$A^3 = -5(-5A - 6I) - 6A \quad \text{(from (1))}$$

$$A^3 = 25A + 30I - 6A$$

$$A^3 = 19A + 30I$$

Example 7. a. *Let u and v be the eigen vectors of A corresponding to the eigen values 1 and 3 respectively. Prove that u + v is not an eigenvector of A.*

b. *Let A and B be real matrices such that the sum of each row of A is 1 and the sum of each row of B is 2. Then show that 2 is an eigenvalue of AB.*

Solution. a. Given u and v are eigenvectors of A corresponding to the eigenvalues 1 and 3 respectively

So, $Au = 1u$

$Av = 3v$

Now, $A(u + v) = Au + Av$

$= 1u + 3v$

So, $(u + v)$ is not an eigenvector of A.

b. Try yourself.

Example 8. *Let A be a* 3×3 *real non-diagonal matrix with* $A^{-1} = A$. *Show that* $tr(A) = -det(A) = \pm 1$.

Solution. Given, $A^{-1} = A \quad \Rightarrow \quad A^2 = I$

So, all the eigenvalues of A^2 are 1

Also, $A^2 - I = 0$

or, $(A - I)(A + I) = 0$

So, two eigenvalues of A are +1 and –1

Since, eigenvalues of A^2 are square of eigenvalues of A.

So, eigenvalues of A will either +1 or –1

So, the third eigenvalue can be +1 or –1

Determinant of matrix is equal to product of its eigenvalues
So, determinant can be ±1
If third eigenvalue is +1, tr $(A) = 1$, det $(A) = -1$
If third eigenvalue is –1, tr $(A) = -1$, det $(A) = 1$
So, tr $(A) = -\det(A) = \pm 1$.

SUMMARY

1. Matrix is an array representation of $(m \times n)$ elements and write as
$$A = [a_{ij}]_{m \times n},\ i = 1, 2, \ldots, m$$
$$j = 1, 2, \ldots, n$$
2. **Rank:** Number of non-zero rows in an echelon form of the matrix is called the rank of matrix.
3. For a non-homogeneous system
   a. If $\rho(A) = \rho(A : B)$ = number of unknowns, then the system is consistent with unique solution.
   b. If $\rho(A) = \rho(A : B)$ < number of unknowns, then system is consistent with infinite many solutions.
   c. If $\rho(A) \neq \rho(A : B)$, then system is inconsistent.
4. For a homogeneous system
   a. If $\rho(A)$ = number of unknowns, then system is consistent with unique solution (trivial solution).
   b. If $\rho(A)$ < number of unknowns, then system is consistent with infinite solutions (non-trivial solutions).
5. If the determinant of a matrix is non-zero, then its inverse exists and it is always unique.
6. Let A be a square matrix of order n over a field F, if $\exists$ a non-zero column vector $X \in F^n$ such that $AX = \lambda X$ for some $\lambda \in F$, then X is called the Eigen vector of A corresponding to λ and λ is called an eigen value of A corresponding to X.
7. Sum of eigen values = Trace (A)
   Product of eigen values = det. (A)
8. A $n \times n$ matrix is diagonalizable iff it has n linearly independent eigen vectors.
9. **Cayley-Hamilton Theorem:** Every square matrix satisfies its own characteristic equation.

OBJECTIVE QUESTIONS

1. If $2\begin{bmatrix} x & 9 \\ y & 6 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$, then the values of x and y are
   a. $x = 6, y = 3$ b. $y = 6, x = 3$ c. $x = 9/2, y = 6$ d. $x = -1, y = -2$
2. If $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then the value of the matrix X is,
   a. $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$ c. $\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ d. $\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$

3. Let A be a square matrix of order 3, then $|kA|$ is equal to

a. $3k|A|$ b. $k|A|$ c. $k^2|A|$ d. $k^3|A|$

4. If $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$, then value of x is

a. 3 b. 4 c. 2 d. –1

5. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

a. 0 b. det A c. 1 d. $\dfrac{1}{\det A}$

6. The value of the determinant $\begin{vmatrix} 0 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$ is

a. 1 b. –1 c. 0 d. 2

7. If $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$, then the value of x is,

a. 7 b. 8 c. 12 d. 10

8. If a matrix A is both symmetric and skew-symmetric, then

a. A is a diagonal matrix b. A is a zero matrix
c. A is a scalar matrix d. A is a square matrix

9. If A $= \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ and $a^2 + b^2 + c^2 + d^2 = 1$, then A^{-1} is

a. $\begin{bmatrix} a+ib & -c+id \\ -a+id & a-ib \end{bmatrix}$ b. $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$ c. $\begin{bmatrix} a-ib & c-id \\ -c-id & a+ib \end{bmatrix}$ d. $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$

10. The rank of matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$, a, b, c being real, is 3, then

a. $a = b = c$
b. a, b, c are all different but $a + b + c = 0$
c. two of the numbers a, b, c are equal but are different from the third
d. a, b, c are all different and $a + b + c \neq 0$

11. Find the value of p for which, the rank of the given matrix is 1.

$$\begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$$

a. 4 b. 2 c. 3 d. 1

12. Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$ be expressed as $P + Q$, where P is symmetric matrix and Q is skew-symmetric matrix, which one of the following is correct?

a. $Q = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 0 \end{bmatrix}$ c. $Q = \frac{1}{2}\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ d. $Q = \begin{bmatrix} 0 & 3/2 \\ 3/2 & 0 \end{bmatrix}$

13. The columns of an orthogonal matrix form

a. an orthogonal set of vectors b. an orthonormal set of vectors

c. a linearly independent set d. All of the above

14. A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1, -1)^T$ and $(2, 1)^T$, respectively, then, M is

a. $\begin{bmatrix} -4 & -8 \\ 5 & 9 \end{bmatrix}$ b. $\begin{bmatrix} 9 & -8 \\ 5 & -4 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ d. $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

15. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, then the modal matrix P is

a. $\begin{bmatrix} -1 & 1 \\ 1+i & 1-i \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 \\ 1-i & 1+i \end{bmatrix}$ c. $\begin{bmatrix} 1 & -1 \\ 1+i & 1-i \end{bmatrix}$ d. $\begin{bmatrix} -1 & -1 \\ 1-i & 1+i \end{bmatrix}$

16. If the characteristic roots of $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ are λ_1 and λ_2, then the characteristic roots of $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ are

a. $\lambda_1 + \lambda_2, \lambda_1 - \lambda_2$ b. $2\lambda_1$ and $2\lambda_2$ c. $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ d. $\lambda_1 + \lambda_2$ and $|\lambda_1 - \lambda_2|$

17. If $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which of the following is zero matrix?

a. $A^2 - A - 5I$ b. $A^2 + A - 5I$ c. $A^2 + A - \mathrm{I}$ d. $A^2 - 3A + 5I$

18. The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ are

a. 1, 4 b. –1, 2 c. 0, 5 d. 2, –5

19. Which one of the following is an eigen vector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$?

a. $[1 \quad -2 \quad 0 \quad 0]^T$ b. $[0 \quad 0 \quad 1 \quad 0]^T$ c. $[1 \quad 0 \quad 0 \quad -2]^T$ d. $[1 \quad -1 \quad 2 \quad 1]^T$

20. The eigen vector of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?

a. 0 b. 1/2 c. 1 d. 4

21. An eigen vector of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

a. $[-1 \; 1 \; 1]^T$ b. $[1 \; 2 \; 1]^T$ c. $[1 \; -1 \; 2]^T$ d. $[2 \; 1 \; -1]^T$

22. Consider the following matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigen values of A are 4 and 8, then

a. $x = 4, y = 10$ b. $x = 5, y = 8$ c. $x = -3, y = 9$ d. $x = -4, y = 10$

23. For the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$, one of the eigen value is 3. The other two eigen values are

a. 2, –5 b. 3, –5 c. 2, 5 d. 3, 5

24. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then the eigen values of A are

a. 2, 1, 0 b. 2, $(1 + i)$, $(1 - i)$ c. 2, –1, –1 d. 1, –1, 0

25. Which of the following matrix is not diagonalizable?

a. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Answers

1. b	**2.** c	**3.** d	**4.** c
5. d	**6.** c	**7.** d	**8.** b
9. c	**10.** d	**11.** c	**12.** c
13. d	**14.** d	**15.** b	**16.** c
17. c	**18.** c	**19.** b	**20.** b
21. b	**22.** d	**23.** b	**24.** c
25. c			

SUBJECTIVE UNSOLVED QUESTIONS (HOTS)

1. Show that if A has a zero row, then AB also has a zero row.
2. Show that if B has a zero column, then AB also has a zero column.
3. Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.
4. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exist a, 3×1 matrix B and 1×3 matrix C, such that $A = BC$.
5. Find 2×2 invertible matrices A and B such that $A + B$ is not equal to zero and $A + B$ is not invertible.
6. Let $A \in M_{n\times n}(F)$. Under what conditions, det $(-A)$ = det (A).
7. Give a counter example to the following statement: If the co-efficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

8. Determine the values of a, b and c so that $(1, 0, -1)$ and $(0, 1, -1)$ are eigen vectors of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}$$

9. Let P, D and A be real square matrices of same order such that P is invertible, D is diagonal and $D = PAP^{-1}$. If $A^n = 0$ for some $n \in N$, then show that $A = 0$.

10. Let A be an $n \times n$ real symmetric matrix with n distinct eigen values. Prove that there exists an orthogonal matrix P such that $AP = PD$ where D is a real diagonal matrix.

11. Find the number and exhibit all 2×2 orthogonal matrices of the form $\begin{bmatrix} 1/3 & x \\ y & z \end{bmatrix}$.

12. Find a 3×3 orthogonal matrix P whose first two rows are multiples of:

a. $(1, 2, 3)$ and $(0, -2, 3)$ b. $(1, 3, 1)$ and $(1, 0, -1)$

13. Let A be a real skew-symmetric matrix, that is, $A^T = -A$. Then prove the following statements.

a. Each eigenvalue of the real skew-symmetric matrix A is either 0 or a purely imaginary number.

b. The rank of A is even.

14. Prove that if $A \in M_{n \times n}(F)$ has n distinct eigenvalues, then A is diagonalisable.

15. Prove that two distinct eigenvectors corresponding to the same eigenvalues are always linearly dependent.

16. For the given 2×2 matrix

$$A = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}$$

a. Find the eigen values of A:

b. For each eigenvalue of A, determine the eigenvector.

c. Diagonalise the matrix A.

d. Using the result of the diagonalisation, compute and simplify A^k for each positive integer k.

PROJECT/PRACTICAL/ACTIVITIES

PROJECT

1. Prepare a model showing various types of matrices and their applications in graph theory.
2. "Diagonalisation helps in determining powers of A *i.e.* A to the power n, where n is a general integer." Explain mathematically as well as with the help of an example.

PRACTICAL

1. Write a MATLAB function that takes a matrix, a row number and a column number. Beginning with the row number passed to the function, scroll down the column passed to the function and return the row number that contains the largest absolute value in the column.
2. Using MATLAB, find the determinant of the 3×3 matrix.

3. Implement the power method (with normalization), for computing eigen values and eigenvectors of a matrix $A \in R_{n\times n}$ in MATLAB.

ACTIVITY

1. A shopkeeper sells packets P_1 of 1 kg. of wheat, 1 kg of rice and 1 kg of Bajra and P_2 containing of 1 kg. of wheat, 0 kg of rice and 1 kg of Bajra and P_3 comprising of 0 kg. of wheat, 1 kg of rice and 1 kg of Bajra.
   Check, Is it possible to buy only one kg. of Bajra?
   If Yes, How?
2. Form a group of students from various cities, make a graph and form adjacency matrix for the same with vertices as cities and edges as transportation cost of a good.
3. Explain how a linear system of differential equations $(dx/dt) = X$, where X is a $m \times m$ diagonal matrix with constant entries, can be solved using the diagonalisation concept.

KNOW MORE

1. The least value of the product xyz for which the determinant $\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix}$ is non-negative is

   a. -8 b. -1 c. $-2\sqrt{2}$ d. $-16\sqrt{2}$

2. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$, then k is equal to

   a. $\alpha\beta$ b. $1/\alpha\beta$ c. 1 d. -1

3. Find the rank of $A = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$.

4. Find 'a' so that the rank of the matrix $A = \begin{bmatrix} a & 0 & 1 \\ 1 & 2 & a \\ 1 & 2 & 3 \end{bmatrix}$ is less than 3.

5. If A is a (2×2) matrix over R with Det $(A + I) = 1 +$ Det (A), then we can conclude that

   a. Det $(A) = 0$ b $A = 0$ c. Tr $(A) = 0$ d. A is non-singular

6. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$, then calculate A^9.

   a. $511A + 510I$ b $309A + 104I$ c. $154A + 155I$ d. exp. $(9A)$

7. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then A^{50} is

a. $\begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 \\ 48 & 1 & 0 \\ 48 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix}$

Answers

1. a **2.** c

3. $\rho(A) = \begin{cases} 3 & \text{if } x \neq y, y \neq z, z \neq x \\ 2 & \text{if either } x = y \quad \text{or } x = z \text{ and } y \neq z \\ 1 & \text{if } x = y = z \end{cases}$ **4.** 0, 3

5. c **6.** a **7.** c

REFERENCES/SUGGESTED READINGS

1. Dass, H.K. Advanced Engineering Mathematics, S. Chand Publications.
2. Dettman, J.W. (1974). Introduction to Linear Algebra and Differential Equations, McGraw Hill, Kogakusha.
3. Garg, Reena (2019). Engineering Mathematics-I, 2nd Edition, Khanna Book Publishing Co. (P) Ltd.
4. Grewal, B.S. Higher Engineering Mathematics, Khanna Publishers
5. Herstein, I.N. (1993). Topics in Algebra, Wiley Eastern.
6. Hohn, E.Franz (1964). Elementary Matrix Algebra, Macmillan Company, New York.
7. Jain, R.K.; Iyengar, S.R.K. Advanced Engineering Mathematics, 2nd Edition , Narosa.
8. Philip, Franklin (1940). A Treatise on Advanced Calculus, Wiley, Inc. New York.
9. Prasad, Chandrika (1967). Mathematics for Engineers, Pothishala Private Ltd.
10. Ram, Babu. Engineering Mathematics, Pearson.
11. Thomas, G.B. and Finney, R.L. (1992). Calculus and Analytic Geometry, Addision-Wesley, Reading, MA.

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CO AND PO ATTAINMENT TABLE

Course outcomes (COs) for this course can be mapped with the programme outcomes (POs) after the completion of the course and a correlation can be made for the attainment of POs to analyze the gap. After proper analysis of the gap in the attainment of POs necessary measures can be taken to overcome the gaps.

Table for CO and PO attainment

Course Outcomes	**Attainment of Programme Outcomes** *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*											
	PO-1	**PO-2**	**PO-3**	**PO-4**	**PO-5**	**PO-6**	**PO-7**	**PO-8**	**PO-9**	**PO-10**	**PO-11**	**PO-12**
CO-1												
CO-2												
CO-3												
CO-4												
CO-5												
CO-6												

The data filled in the above table can be used for gap analysis.